The symmetric orbifold from the world-sheet

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Based on work with Lorenz Eberhardt, Kevin Ferreira, Rajesh Gopakumar, Chris Hull, and Juan Jottar.

Motivation

It is generally believed that the CFT dual of string theory on

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

is on the same moduli space of CFTs that also contains the symmetric orbifold theory

$$\mathrm{Sym}_N(\mathbb{T}^4) \equiv (\mathbb{T}^4)^N / S_N$$

[Maldacena ’97], …. see e.g. [David et.al. ’02]

Motivation

However, it is not known what precise string background is being described by the symmetric orbifold theory itself.

see however [Larsen, Martinec '99]

On the other hand, there is an explicitly solvable world-sheet theory for strings on this background in terms of an sl(2,R) WZW model.

[Maldacena, (Son), Ooguri '00 & '01]

However, it is not known what precise dual CFT (on the above moduli space) this corresponds to.

Motivation

In fact, the only consensus was that the actual symmetric orbifold theory **cannot** be dual to the WZW model…

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In fact, the only consensus was that the actual symmetric orbifold theory **cannot** be dual to the WZW model…

The basic reason for this is that the WZW model describes the background with pure NS-NS flux, which is known to have long string solutions.

[Seiberg, Witten ’99], [Maldacena, Ooguri ’00]

These long strings live near the boundary of AdS, and they give rise to a continuum of excitations that are not present in the actual symmetric orbifold theory.

Higher Spins

In a separate development, the higher spin version of the AdS/CFT duality was studied.

At the tensionless point in moduli space, string theory on AdS is dual to a (nearly) free conformal field theory.

The conserved currents of the free CFT correspond to massless higher spin fields in AdS, and the tensionless string theory contains a Vasiliev higher spin theory as a (closed) subsector.

[Fradkin & Vasiliev, ’87]
[Vasiliev, ’99…]

[Sundborg, ’01], [Witten, ’01], [Mikhailov, ’02],
[Klebanov & Polyakov, ’02], [Sezgin & Sundell, ’03..]

HS theory — CFT duality

Concrete realisation of this idea in context of AdS_3 : there exists a HS AdS/CFT duality of the form

[MRG, Gopakumar '13 & '14]

$\boxed{\text{large } \mathcal{N} = 4}$

hs theory based on

$$\mathrm{shs}_2[\lambda] \quad \longleftrightarrow \quad \frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N+4)_1}{\mathfrak{su}(N)_{k+2} \oplus \mathfrak{u}(1)_\kappa} \oplus \mathfrak{u}(1)_\kappa \ .$$

Wolf space cosets

[Sevrin, Troost, Van Proeyen, Schoutens, Spindel, .. '88/'89]

in 't Hooft limit with $\lambda = \dfrac{N+1}{N+k+2}$.

hs theory in string theory

and it embeds naturally into stringy duality as

[MRG, Gopakumar '14]

$\boxed{\text{large } \mathcal{N} = 4}$

$\boxed{\text{small } \mathcal{N} = 4}$

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 \xrightarrow{\lambda \to 0} \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

hs theory based on

$$\mathrm{shs}_2[\lambda]$$

$\updownarrow$

string theory

$\updownarrow$

Wolf space cosets $\subset$ symmetric orbifold

$$\frac{\mathfrak{su}(N+2)_k \oplus \mathfrak{so}(4N+4)_1}{\mathfrak{su}(N)_{k+2} \oplus \mathfrak{u}(1)_\kappa} \oplus \mathfrak{u}(1)_\kappa \ .$$

$$\mathrm{Sym}_{N+1}(\mathbb{T}^4) \equiv (\mathbb{T}^4)^{\otimes(N+1)}/S_{N+1}$$

Symmetric orbifold

In particular, this line of reasoning suggests that the symmetric orbifold theory should correspond to a tensionless limit of string theory on AdS3.

The tensionless limit arises when the spacetime geometry is of string size, i.e. in the deep stringy regime.

In the context of the WZW description, this should be the situation where the level of the sl(2,R) affine theory takes the smallest possible value.

WZW model

This led us to study the spacetime spectrum of the k=1 sl(2,R) WZW systematically.

As will be explained in more detail below, we found that the k=1 theory indeed has massless higher spin fields, and that its spectrum resembles that of the symmetric orbifold theory in the large N limit.

[MRG, Gopakumar, Hull ’17],
[Ferreira, MRG, Jottar ’17],
[MRG, Gopakumar ’18]
see also [Giribet et.al. ’18]

WZW model

However, the k=1 theory in the NS-R formalism is not really well-defined.

In particular, the full WZW model is in this case

$$\mathfrak{sl}(2)_k^{(1)} \oplus \mathfrak{su}(2)_k^{(1)} \oplus [\mathfrak{u}(1)^{(1)}]^{\oplus 4}$$

and at k=1

$$\mathfrak{su}(2)_1^{(1)} \cong \mathfrak{su}(2)_{-1} \oplus 3 \text{ free fermions}$$

non-unitary

WZW model

Furthermore, the WZW model still seems to contain a continuum of states (that are not present in the symmetric orbifold theory).

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Furthermore, the WZW model still seems to contain a continuum of states (that are not present in the symmetric orbifold theory).

As it turns out, both of these problems can be overcome by considering the alternative description of string theory on AdS3 x S3 in terms of the so-called hybrid formalism.

[Berkovits, Vafa, Witten ’99]

WZW model

In this formulation, the AdS3 x S3 part is described (for pure NS-NS flux) by a supergroup WZW model, namely

$$\mathfrak{psu}(1,1|2)_k$$

and this description continues to make sense also for k=1. However, something special happens for this value: as will be explained below, the representation theory is much more constrained for k=1, and in particular, the continuum of representations is not allowed any longer.

[Eberhardt, MRG, Gopakumar ’18]

WZW model

Taking this into account, we have shown that the resulting spacetime spectrum agrees precisely with that of the symmetric orbifold theory (in the large N limit)!

[Eberhardt, MRG, Gopakumar '18]

In fact, the resulting theory shows strong signs of being a `topological' string. Furthermore, it has a free field realisation, reflecting the essentially free nature of the dual symmetric orbifold.

Plan of talk

1. **Introduction and Motivation**
2. The NS-R construction
3. The supergroup hybrid formulation
4. Conclusions and Outlook

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NS-R WZW model

Let us begin by reviewing some basic facts about the WZW model based on sl(2,R). [Maldacena, Ooguri '00]

In the susy case, the relevant chiral algebra is

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus 3 \text{ free fermions}$$

bosonic: $J^3_n, J^\pm_n$

decoupled

The free fermions sit in the usual NS/R representations.

NS-R WZW model

The representations of the bosonic sl(2,R) affine algebra are characterised by the sl(2,R) reps of the highest weights. There are 2 classes of sl(2,R) reps that appear:

[Maldacena, Ooguri '00]

Discrete lowest weight reps:

$$\mathcal{D}_j^+ : \qquad C = -j(j-1) \ , \quad J_0^- |j,j\rangle = 0$$

Continuous reps:

$$C_\alpha^j : \quad C = -j(j-1) = \tfrac{1}{4} + p^2 \ , \quad |j,m\rangle \text{ with } m \in \alpha + \mathbb{Z}$$

$$(j = \tfrac{1}{2} + ip)$$

No-ghost theorem

Because of the Maldacena-Ooguri (unitarity) bound,

$$\text{MO-bound}: \qquad \frac{1}{2} < j < \frac{k+1}{2}$$

[Petropoulos ’90]
[Hwang ’91]
[Evans, MRG, Perry ’98]
[Maldacena, Ooguri ’00]

the (discrete) spectrum is bounded from above. Additional states are spectrally flowed images of these two classes of representations

They are not Virasoro highest weight, and are therefore best described in terms of the spectral flow w.

[Maldacena, Ooguri ’00]
see also [Henningson et.al. ’91]

Spectral flow automorphism

Basic idea: work with original representation space, but define on it a new action (by automorphism):

$$
\begin{aligned}
\hat{J}_n^{\pm} &\equiv \alpha_w(J_n^{\pm}) = J_{n\mp w}^{\pm} \\
\hat{J}_n^{3} &\equiv \alpha_w(J_n^{3}) = J_n^3 + \tfrac{k}{2} w \delta_{n,0} \qquad\qquad (w \in \mathbb{N}) \\
\hat{L}_n &\equiv \alpha_w(L_n) = L_n - w J_n^3 - \tfrac{k}{4} w^2 \delta_{n,0} \ .
\end{aligned}
$$

Since the automorphism is outer, get a new representation in this manner: spectrally flowed rep.

Physical states

This description is covariant, i.e. we need to impose the physical state condition, e.g. in NS sector

$$G_r^{\mathrm{tot}}\Phi = 0 \quad (r > 0)$$
$$(L_0^{\mathrm{tot}} - \tfrac{1}{2})\Phi = 0 \ .$$

In particular, the second condition (mass-shell) condition implies that

$$\frac{C}{k} + h_0 + N = \frac{1}{2} \qquad (\text{NS-sector})$$

Casimir of sl(2,R)

World-sheet conformal dim. of internal CFT

Dual CFT

The dual (`spacetime') CFT lives on the boundary of AdS3, and we have the identifications

$$L_0^{\rm CFT} = J_0^3 \ , \qquad L_1^{\rm CFT} = J_0^- \ , \qquad L_{-1}^{\rm CFT} = J_0^+ \ ,$$

with a similar relation for the right-movers.

With these preparations at hand, we can now study the physical spectrum of the (spacetime) theory for k=1.

As we shall see, the interesting part of the spectrum comes from the spectrally flowed continuous reps.

Continuous reps

For the spectrally flowed continuous reps, the mass-shell condition (in the NS sector) is at k=1

$[\frac{C}{k} + h_0 + N = \frac{1}{2}]$

$[\alpha_w(L_n) = L_n - wJ^3_n - \frac{k}{4}w^2\delta_{n,0}]$

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{where } C = \frac{1}{4} + p^2$$

Here m is the J^3_0 eigenvalue before spectral flow, and we have set $h_0 = 0$ (for simplicity).

For the continuous representations we can simply solve this equation for m. For the case of p=0 we then get

Continuous reps

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{with } C = \tfrac{1}{4}$$

$$m = \frac{1}{w}\left[N - \frac{w^2+1}{4}\right]$$

Continuous reps

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{with } C = \tfrac{1}{4}$$

$$m = \frac{1}{w}\left[N - \frac{w^2+1}{4}\right]$$

Then observing that the actual J_0^3 eigenvalue is

$[\alpha_w(J_n^3) = J_n^3 + \frac{k}{2} w \delta_{n,0}]$

$$h = m + \frac{w}{2} = \frac{N}{w} + \frac{w^2-1}{4w} \; .$$

Full spectrum

[MRG, Gopakumar '18]
see also [Giribet, et.al. '18]

$$h = m + \frac{w}{2} = \frac{N}{w} + \frac{w^2 - 1}{4w} \,.$$

w-twisted modes

ground state energy in w-twisted sector

Symmetric orbifold formula for cycle length w!

Note that for w=1 and N=0, this includes in particular chiral states (h=0) that correspond to massless higher spin fields!

[MRG, Gopakumar, Hull '17]
[Ferreira, MRG, Jottar '17]]

Which orbifold

For $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ at k=1, criticality implies that the bosonic su(2) factor appears at level -1, and thus the analysis in the NS-R sector is a bit formal — in the hybrid formalism this will be cleaner (see below).

In order to get a sense of what will happen, we can use that

$$\mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1) = 4 \text{ symplectic bosons}$$

[Goddard, Olive, Waterson '87]

Which orbifold

The 4 symplectic bosons behave as ghosts (on the level of the partition function) and remove 4 of the 8 fermions.

This therefore suggests that we end up with 4+4 free bosons and fermions, i.e. with the spectrum of

$$\boxed{\text{symmetric orbifold of } \mathbb{T}^4}.$$

[MRG, Gopakumar '18]

Continuum of states

However, the spectrum still seems to have a continuum (we earlier set p=0 by hand), which is not present in the symmetric orbifold theory.

There are also some discrete rep states that do not fit into the above.

And the above treatment of $\mathfrak{su}(2)_{-1}$ was a bit formal…

Thus we have not quite managed yet to identify the world-sheet theory that corresponds to the symmetric orbifold.

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Hybrid formalism

In the hybrid formalism the world-sheet theory is described (for pure NS-NS flux) by the WZW model based on

$$\mathfrak{psu}(1,1|2)_k$$

together with the (topologically twisted) sigma model for T4. For generic k, this description agrees with the NS-R description a la MO.

[Troost '11], [MRG, Gerigk '11]
[Gerigk '12]

Hybrid formalism

For the following it will be important to understand the representation theory of

$$\mathfrak{psu}(1,1|2)_1$$

The bosonic subalgebra of this superaffine algebra is

$$\mathfrak{sl}(2)_1 \oplus \mathfrak{su}(2)_1$$

Thus only **n**=1 and **n**=2 are allowed for the highest weight states.

Hybrid formalism

One of the key differences to the NS-R formalism is that the fermions of

$$\mathfrak{psu}(1,1|2)_1$$

do not sit in the adjoint representation of the bosonic subalgebra, but rather in bispinor representations.

As a consequence, one cannot decouple the fermions as before and therefore obtain a negative level for su(2). In fact, the k=1 theory seems to be well-defined.

[Eberhardt, MRG, Gopakumar ’18]

Short representations

A generic representation of the zero mode algebra $\mathfrak{psu}(1,1|2)$ has the form

$$(C^j_\alpha, \mathbf{n})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^{j+1}_\alpha, \mathbf{n}) \qquad (C^j_\alpha, \mathbf{n+2}) \qquad 2\cdot(C^j_\alpha, \mathbf{n}) \qquad (C^j_\alpha, \mathbf{n-2}) \qquad (C^{j+1}_\alpha, \mathbf{n})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^j_\alpha, \mathbf{n})$$

continuous rep
of sl(2,R)

rep of su(2) of
dim = n+1.

Short representations

A generic representation of the zero mode algebra $\mathfrak{psu}(1,1|2)$ has the form

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$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^{j+1}_\alpha, \mathbf{n}) \quad (C^j_\alpha, \mathbf{n+2}) \quad 2\cdot(C^j_\alpha, \mathbf{n}) \quad (C^j_\alpha, \mathbf{n-2}) \quad (C^{j+1}_\alpha, \mathbf{n})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^j_\alpha, \mathbf{n})$$

continuous rep
of sl(2,R)

Thus for k=1 need
a short rep!

Short representations

In fact, the only representations that are allowed are

$$(C^{j}_{\alpha}, \mathbf{2})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1})$$

and the shortening condition actually implies that this is only possible provided that

$$j = \frac{1}{2} \quad \longrightarrow$$

NO CONTINUUM!

[Eberhardt, MRG, Gopakumar ’18]

Short representations

The corresponding affine representation has in fact many null-vectors, and thus, after including the ghosts, the contribution from

$$\mathfrak{psu}(1,1|2)_1$$

just reduces to the zero-modes (which are fixed by the mass-shell condition): topological sector!

One also finds that these are the only representations, i.e. no discrete representations appear.

Short representations

[Eberhardt, MRG, Gopakumar '18]

For these representations, the partition function localises to isolated points of the world-sheet modular integral

$$\sum_{m\in\mathbb{Z}} \delta(t - \tau w + m) \ .$$

space-time modular parameter

world-sheet modular parameter

These are precisely the points where the world-sheet torus can be mapped holomorphically to the boundary torus — reminiscent of A-model…

see also [Maldacena, Ooguri '01]

Physical spectrum

The rest of the analysis works essentially as in the NS-R formulation. Because now the continuum has disappeared (and there are no discrete representations) we get exactly the (single-particle) spectrum of

[Eberhardt, MRG, Gopakumar ’18]

$$\mathrm{Sym}_N(\mathbb{T}^4)$$

where, as before, the spectral flow parameter w is to be identified with the length of the single cycle twisted sector in the symmetric orbifold (in the large N limit).

Free field realisation

The affine algebra $\mathfrak{psu}(1,1|2)_1$ actually has a free field realisation in terms of

[Eberhardt, MRG, Gopakumar '18]

$$\begin{aligned}\mathfrak{psu}(1,1|2)_1 &\cong \frac{\mathfrak{u}(1,1|2)_1}{\mathfrak{u}(1)_U \oplus \mathfrak{u}(1)_V} \\ &\cong \frac{\text{2 pairs of symplectic bosons and 2 complex fermions}}{\mathfrak{u}(1)_U \oplus \mathfrak{u}(1)_V} \ .\end{aligned}$$

This allows us to calculate the characters, the fusion rules, etc., and show (with some effort) that the world-sheet theory is consistent.

see also [Gotz, Quella, Schomerus '06]
[Ridout '10]

Fusion rules

We have also managed to show how the (single-particle) symmetric orbifold fusion rules arise from the world-sheet.

The main subtlety has to do with the fact that in order to analyse the fusion rules of the spacetime theory, we need to work in the so-called x-basis of the world-sheet theory.

[Eberhardt, MRG, Gopakumar '18]
see also [Maldacena, Ooguri '01]

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Conclusions and Outlook

We have given strong evidence that the large N limit (= weak string coupling) of the symmetric orbifold theory is exactly dual to string theory with one unit of NS-NS flux (k=1):

$$\mathrm{Sym}_N(\mathbb{T}^4) \quad = \quad \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

1 unit of NS-NS flux

This background describes a tensionless string theory, where massless higher spin fields are present.

Conclusions and Outlook

$$\mathrm{Sym}_N(\mathbb{T}^4) \quad = \quad \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

1 unit of NS-NS flux

Both sides are explicitly solvable and have free field realisations.

This opens the door for all sorts of quantitative tests of the (stringy) duality. It may also allow one to prove the duality in this case.

Conclusions and Outlook

$$\mathrm{Sym}_N(\mathbb{T}^4) \quad = \quad \mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$$

1 unit of NS-NS flux

The world-sheet theory exhibits signs of a topological string theory:

- only short representations of psu(1,1|2) appear
- modular integral localises to holomorphic maps

cf [Aharony, David, Gopakumar, Komargodski, Razamat ’07]
[Razamat ’08], [Gopakumar ’11], [Gopakumar, Pius ’12]

Future directions

Many directions for future work:

- generalise analysis to K3 and S3 x S1

  [Eberhardt, MRG, in progress]

- understand topological structure directly

- check further aspects of correspondence, e.g. correlators, Euclidean path integral, …
- prove by some sort of field redefinition

- study deformations…

Thank you!