Symmetry, Representation, Inversion Formula in Galilean Conformal Theories

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Motivation

- Holographic duality beyond AdS/CFT: Newton-Carton like $AdS_2 \times R^1/GCA_2$; TMG_3/GCA_2; BMS_3/GCA_2.[Bagchi, 09 '; Bagchi, 10 ']
- Galilean conformal theories can be realized as the non-relativistic limit of the CFT_2 [Bagchi, Gopakumar, Mandal, Miwa 09 ']
- Are there any concrete Galilean conformal field theories other than free theories or by taking the non-relativistic limit?
- Studying the bootstrap by inversion formula may give some clues.
- Algebra structure: The algebra is the special semi-direct sum of Virasoro algebra and $U(1)$. The global algebra is not semi-simple. But it is simpler than other cases. This will promote and enlarge the discussions on bootstrap.

Introduction

In 2D Quantum field theories, the global symmetries of translations as well as dilation of one direction are enhanced to two (minimal) sets of infinite dimensional algebras.[Hofman, Strominger, 11 ']

$$Vir \times Vir \qquad\qquad Vir - Kac - Moody$$

What about general Lifshitz scalings?

2D Lifshitz Scaling

- The global symmetries we consider are

$$H : x \to x' = x + \delta x$$

$$\bar{H} : y \to y' = y + \delta y$$

$$D : x \to x' = \lambda^a x, \qquad y \to y' = \lambda^b y$$

- The dilation scales the two direction at the same time, but with different weights.
- We also consider the theories defined on a Newton-Cartan Geometry, so that the Galilean boost serves at least as a local symmetry.

$$B : y \to y' = y + vx$$

2D Lifshitz Scaling

- Assumptions: locality, discrete non-negative dilation spectrum, which can be diagonizable, discrete boost spectrum.
- There exists a complete basis of local operators so that

$$[H, O] = \partial_x O, \quad [\bar{H}, O] = \partial_y O, \quad [B, O] = x\partial_y O + \xi O$$

$$[D, O] = ax\partial_x O + by\partial_y O + \Delta_O O$$

- The currents can be shifted by adding local operators without changing the canonical commutation relations.

2D Lifshitz Scaling

- The x-component of the conserved currents of translation symmetry of y direction $\bar{h}_x$ can be redefined so that it depends on x only, so there is one set of infinite conserved charges related with the Galilean boost,

$$P_\epsilon = \int \epsilon(x)\bar{h}_x(x)dx$$

- Special case: a=0.
  To make the canonical commutation relations hold, one must have

$$\bar{h}(x) = \bar{h}(y) = 0$$

  The theories are actually 1D translational-invariant theories.

2D Lifshitz Scaling

- $a \neq 0$: considering the conservation law of dilation current d, as well as the x-translation current h, there is another set of infinite conserved charges.

$$Q_\epsilon = \int \{a\epsilon(x)h_x(x,y) + b\epsilon\,'(x)y\bar{h}_x(x)\}dx + \int \{a\epsilon(x)h_y(x)\}dy$$

  $\epsilon(x)$ is an arbitrary smooth function on x. $h_y(x)$ depends on x only, since its boost charge vanishes.

- The conserved charges Q_ϵ act on the local operators as,

$$[Q_\epsilon, O(x,y)] = (a\epsilon\partial_x + b\epsilon\,'y\partial_y + a\epsilon\,'\Delta + b\epsilon\,'\,'\xi)O(x,y)$$

2D Lifshitz Scaling

- Algebra:

$$
\begin{aligned}
[Q_A, Q_B] &= Q_{aA'B-aB'A} + \frac{c_1}{12}\int \{aA''B' - aB''A'\}dx \\
[Q_A, P_B] &= P_{bA'B-aB'A} + \frac{c_2}{12}\int \{bA''B' - bB''A'\}dx \\
[P_A, P_B] &= \frac{k}{2}\int \{(a-b)A'B - (a-b)B'A\}dx
\end{aligned}
$$

- Plane Mode Algebra without central extension:

$$
\begin{aligned}
[Q_n, Q_m] &= (n-m)Q_{n+m} \\
[Q_n, P_m] &= (bn-am)P_{n+m} \\
[P_n, P_m] &= 0
\end{aligned}
$$

which is called the infinite extension of spin-l Galilean algebra, with $l = \frac{b}{a}$.[Henkel, 96 ']

2D Lifshitz Scaling

- The possible kinds of central extension are determined by the Jacobi identity and dependent on $l = \frac{a}{b}$. [Hosseiny, 14 ']
- T-extension is always allowable.

$$[Q_n, Q_m] = (n-m)Q_{n+m} + \frac{c_T}{12} n(n^2-1)\delta_{n+m,0}$$

- B-extension is only allowable for $l = 1$.

$$[Q_n, P_m] = (n-m)P_{n+m} + \frac{c_B}{12} n(n^2-1)\delta_{n+m,0}$$

- M-extension is only allowable for half-integer l or $b = 0$.

$$[P_{n-l}, P_m] = M(-1)^n \frac{n!(2l-n)!}{(2l)!}\delta_{n+m-l,0} \;\; or \;\; [P_n, P_m] = Mn\delta_{n+m,0}$$

- Warped CFT [Detournay et al., 12 '], GCA [Bagchi, 09 '], Schrodinger field theories [Henkel, 93 '] are correspond to $(a, b) = (1, 0), \; (1, 1), \; (2, 1)$.

Galilean Conformal Algebra (GCA)

- Consider the symmetry on the Newton-Cartan geometry,

$$\begin{cases} x \to & f(x) \\ y \to & f'(x)y \end{cases}, \qquad y \to y + g(x)$$

- The plane modes form the GCA,

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c_1}{12}(n^2-1)n\delta_{n+m,0} \\
[L_n, M_m] &= (n-m)M_{n+m} + \frac{c_2}{12}(n^2-1)n\delta_{n+m,0} \\
[M_n, M_m] &= 0
\end{aligned}$$

Galilean Conformal Algebra (GCA)

- Find the subgroup keeping the origin invariant. A set of local operators $O(0)$ correspond to the irreducible highest weight representation, then induce the representation of GCA,[Bagchi, Mandal 09 '; Hijano 18 ']

$$O(x,y) = U^{-1} O(0) U, \qquad U = e^{-xL_{-1} - yM_{-1}}$$

- The primary operators transforms under the symmetry,

$$O'(f(x), f'(x)y + g(x)) = f'(x)^{-\Delta} e^{-\xi(\frac{g'(x)}{f'(x)} + y\frac{f''x}{f'(x)})} O(x,y)$$

where Δ is the weight, ξ is the boost charge.

Gram Determinant and Null States

- We calculate the Gram determinant of arbitrary level N in the Galilean conformal field theories,

$$det\ M_N = (-1)^N \left[\prod_{ab \leq N} c(a,b)^{\sum_{i=0}^{N-ab} P(i) f(N-ab-i,a)} \right]^2$$

$$c(a,b) = \left(2a\xi + \frac{c_2}{12}(a^3 - a)\right)^b b!$$

$P(N)$ is the partition function of integer N, and $f(N,a)$ is the partition function of N while integer a does not appear in the partition.

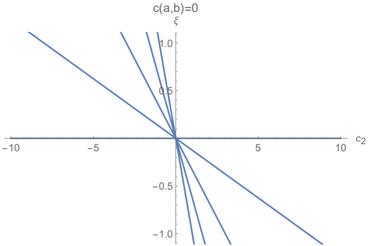
$$\sum_{N=0}^{\infty} f(N,a) x^N = \prod_{k \neq a}^{\infty} \frac{1}{1-x^k}$$

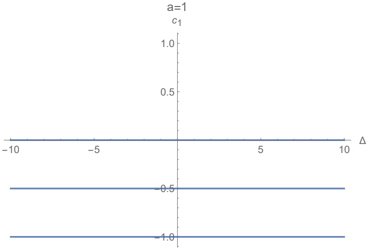
- The Gram determinants are only dependent on ξ and c_2. After ruling out the null states, we find the new Gram determinants dependent on Δ and c_1.

Vanishing curves

- To find potential minimal models, we draw the vanishing curves of the Gram determinant, where there are some null states.

$$c(a,b) = 0, \qquad \Delta + \frac{c_1(a^2-1)}{24} = A(a,b) = const.$$





- Special point: $\xi = c_2 = 0$. This is a chiral CFT_1.
- No other minimal models.

Correlation function

- The correlation functions are determined by the invariance under the global transformation.[Bagchi, Gary, Zodinmawia, 17 '; Bagchi, 09 ']
- The local operators admit a convergent OPE.
- Data: spectrum and OPE coefficients.
- The 4-pt function is

$$G_4(x_{ij}, y_{ij}) = \prod x_{ij}^{-\Delta_{ijk}/3} e^{\frac{y_{ij}}{x_{ij}} \sum \xi_{ijk}/3} G(x, y)$$

  where $\Delta_{ijk} = \Delta_i + \Delta_j - \Delta_k$ and ξ_{ijk} is similar.
- x, y are the global invariant cross ratio,

$$x = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \qquad \frac{y}{x} = \frac{y_{12}}{x_{12}} + \frac{y_{34}}{x_{34}} - \frac{y_{13}}{x_{13}} - \frac{y_{24}}{x_{24}}$$

Crossing equation

- The 4-pt function is invariant under crossing symmetries.
- Under exchange of O_2 and O_4, $x \to 1-x,\ y \to -y$, the invariance of 4-pt function gives the crossing equation.

$$\sum_{\{\Delta,\xi\}}^{s\ channel} P_{\Delta,\xi} g_{\Delta,\xi}(x,y) = \sum_{\{\Delta,\xi\}}^{t\ channel} P_{\Delta,\xi} g_{\Delta,\xi}(1-x,-y)$$

- The operators in Galilean conformal field theories can be organized into Primary + GCA Descendants or Quasi-primary + Global Descendants.
- $\{\Delta,\xi\}$ is primary operators, $\to$ GCA block expansion;
- $\{\Delta,\xi\}$ is quasi-primary, $\to$ global block expansion.

Global Block Expansion

- At level N descendant of a primary operator, all the operators orthogonal to the quasi-primary operators are the global descendants, considering expressing the generators $L_{-n},\ M_{-n}$ as the commutator of $L_{-1},\ L_{-2},\ M_{-1},\ M_{-2}$, e.g.

$$L_{-3} = [L_{-1}, L_{-2}]$$

- However, such quasi-primary operators are generally multiplet of M_0.

$$A = L_{-2}|\Delta, \xi\rangle, \quad B = M_{-2}|\Delta, \xi\rangle$$

$$M_0 \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \xi & 2 \\ 0 & \xi \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

- Reducible but not decomposable.
- Formally, $A = 2\partial_\xi B$.

Global Block Expansion

- The quasi-primary operators have similar behaviour under the action of the Casmir operators.

$$C_2 = M_i M^i = M_0^2 - M_{-1} M_1, \quad C_4 = (L_i M^i)^2$$

- One of the multiplet of rank-r is the eigenvector of the Casmir operators.

$$C_2 O = \lambda_2 = \xi^2 O, \qquad C_4 O = \lambda_4 = 4\xi^2(\Delta - 1)^2 O$$

- For each one of the multiplet,

$$(C_2 - \lambda_2)^\alpha O = 0, \quad (C_4 - \lambda_4)^\alpha O = 0, \quad \alpha = 1, \cdots, r$$

Global Block Expansion

- Inserting a complete set of basis in the four-point functions, one gets the block expansion, which satisfy

$$(C_2 - \lambda_2)^r G^{(r)}_{\Delta,\xi} = 0, \quad (C_4 - \lambda_4)^r G^{(r)}_{\Delta,\xi} = 0$$

- The general solutions are

$$G^{(r)}_{\Delta,\xi} = \sum_{\alpha=0}^{r-1} A_\alpha \partial_\xi^\alpha G^{(0)}_{\Delta,\xi}$$

- The boundary conditions are fixed in the $x \to 0, \ y \to 0$ OPE limit, as well as the Gram matrices.

$$\begin{aligned} O_1 O_2 &= \sum_{\Delta,\xi} C_{\Delta,\xi} x^{-\Delta_1 - \Delta_2 + \Delta} \exp\{(\xi_1 + \xi_2 - \xi)\frac{y}{x}\} \\ &\times \sum_{\{p,q\},a=0}^{p+q} \beta_{\{p,q\},a} x^{p+q} (\frac{y}{x})^a O_{\{p,q\}} \end{aligned}$$

Global Block Expansion

- For $\xi = 0$ cases, SL(2,R) Global Block, since the $M_{-n}|\Delta, 0\rangle$ are null states.
- The global invariant factor of the 4-pt function admits the global block expansion, for $0 < x < 1$

$$\frac{G_4}{G_2 G_2} = \sum_{\{\Delta\}} |_{\xi=0} P_{\Delta,0} x^{\Delta} \; {}_2F_1(\Delta, \Delta, 2\Delta, x) + \sum_{\{\Delta,\xi,\alpha\}} P_{\Delta,\xi,\alpha} P_{\Delta,\xi,\alpha} \partial_\xi^\alpha G^{(0)}_{\Delta,\xi}$$

- $G^{(0)}_{\Delta,\xi}$ is the eigenfunction of C_2, C_4[Bagchi, Gary, Zodinmawia 16 ';17 '].

$$G^{(0)}_{\Delta,\xi} = 2^{2\Delta-2} \frac{x^{\Delta}(1+\sqrt{1-x})^{2-2\Delta}}{\sqrt{1-x}^{\frac{1}{2}}} e^{\frac{-\xi y}{x\sqrt{1-x}}}$$

Why Inversion Formula?

- It is difficult to consider the bootstrap directly from the GCA crossing equation.
- The Lorentz inversion formula simplifies and unifies many studies of the analytic bootstrap.[Caron-Huot, 17 '; Simmons-Duffin, 18 ']
- The LIF gives new conceptual ideas: spin analyticity; representation theory of the Lorentz conformal group.
- GCA is intriguing: NOT semi-simple. "Harmonic analysis"?

What is Inversion Formula?

- Abstract information of OPE coefficient from the 4-pt function.

$$G_4 \to P_{\Delta,\xi}?$$

- A toy model: a function expanded in terms of "blocks" in the region around the origin.[Caron-Huot, 17 ']

$$f(x) = \sum_{J=0}^{\infty} P_J x^J$$

$$f(x) : \textit{4pt Function}, \quad P_J : \textit{Coefficient}, \quad x^J : \textit{Block}$$

- $f(x)$ has branch cuts $(-\infty, -1] \cup [1, \infty)$, and grows no faster than exponential behaviour.

A Toy Model

- Inverse the 4-pt function to get the OPE coefficient.

$$P_J = \frac{1}{2\pi i}\oint_{|x|=c<1} x^{-J-1} f(x)dx \qquad \textit{Euclidean}$$

$$P_J = \frac{1}{2\pi i}\int_1^{\infty} x^{-J-1}(Disc[f(x)]+(-1)^J Disc[f(-x))]dx \quad \textit{Lorentzian}$$

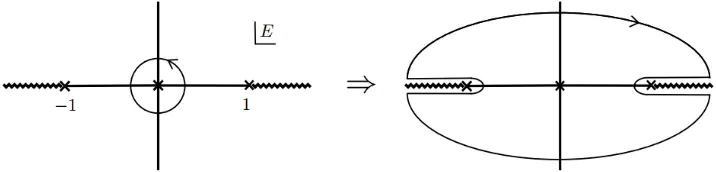


Figure: Contour deformation to get the Lorentzian inversion formula, from [Caron-Huot, 17 ']

A Toy Model

- The Euclidean inversion formula is valid only for integer J.
- Having assumed $|f(x)/x|$ is bounded above (Regge limit), the Lorentz inversion formula is valid for $J > 1$.
- A rigid structure between the coefficients to make the Regge behaviour under control.

$$Finite = Finite + Infinite + Infinite + \cdots \,, \qquad Regge\ limit$$

CFT Inversion formula

- The 4-pt function can be expressed by the integral of principal series rep.[Caron-Huot, 17 ']

$$G_4 = \frac{1}{2\pi i} \sum_{J=0}^{\infty} \int_{d/2-i\infty}^{d/2+i\infty} P(\Delta, J) \Psi_{\Delta,J} d\Delta$$

- The Euclidean inversion formula is

$$P(\Delta, J) = (G_4, \Psi_{d-\Delta,J}), \qquad J \text{ is a integer.}$$

- The Lorentz inversion formula is

$$P(\Delta, J) \sim \int d^2z \mu(z, \bar{z}) G_{J+d-1,\Delta-d+1} dDisc[G_4]$$

- Analyticity in spin.

CFT Bootstrap

- Expand the LIF at large J on the both side, since it is analytic.

$$G_{J+d-1,\Delta-d+1} \sim z^{\frac{J-\Delta}{2}} \bar{z}^{\frac{J+\Delta}{2}}$$

- The dominant part in the integral is $z \to 0,\ \bar{z} \to 1$. The four-point function admits the t channel expansion, where the dominant contribution is from identity operator $(\frac{z\bar{z}}{(1-z)(1-\bar{z})})^{\Delta_O}$.

$$P(\Delta, J) = \frac{\#}{\Delta - J - 2\Delta_O} + \frac{\#}{\Delta - J - 2\Delta_O - 2} + \cdots + O(\frac{1}{J})$$

- Recover the block expansion by contour deformation and picking the poles.

$$P(\Delta, J) \sim \sum_{\{\Delta_0\}} \frac{P_{\Delta,J}}{\Delta - \Delta_0}$$

Key Points on Inversion Formula

- Find the measure to make the Casmir operator Hermitian, so that the eigenfunctions are a complete set of basis. They have different boundary conditions from the blocks.
- The integral over principal series rep. is related to the block expansion by contour deformation (on the Δ plane).
- The Regge bound make it possible to deform the contour on the cross ratio plane, and drop the arc at infinity, giving a analytic structure of the inversion function.

Key Points on Inversion Formula

- In the Regge limit, each individual block grows like $e^{(J-1)t}$. To get the bounded Regge behaviour, Sommerfeld-Watson transform is required: replacing the sum over J with an integral in the imaginary direction.

$$f(x) = \oint \frac{P_J}{1 - e^{-2\pi i J}} x^J dJ$$

- In Euclidean signature, OPE captures the singularities. In Lorentzian signature, there are also Regge poles.
- $SO(d+1,1)$ and $SO(d,2)$ have different principal reps.[Kravchuk, Simmons-Duffin '18]

GCA Inversion Formula

- The block expansion of 4-point functions in s-channel is,

$$\begin{aligned} \frac{G_4}{G_2 G_2} &= \sum_{\{\Delta\}} |_{\xi=0} P_{\Delta,0} x^{\Delta} \ {}_2F_1(\Delta, \Delta, 2\Delta, x) \\ &+ \sum_{\{\Delta,\xi,\alpha\}} P_{\Delta,\xi,\alpha} P_{\Delta,\xi,\alpha} \partial_\xi^\alpha G^{(0)}_{\Delta,\xi} \end{aligned}$$

- There are two indices to label the partial waves, so that two independent Casmir operators are required. The Casmir operators act on the blocks with non-vanishing ξ, as

$$(C_2 - \lambda_2)^r G^{(r)}_{\Delta,\xi} = 0, \quad (C_4 - \lambda_4)^r G^{(r)}_{\Delta,\xi} = 0$$

- However, these Casmir operators cannot select out the $SL(2,R)$ global blocks. In $\xi = 0$ sector, the Casmir is

$$\tilde{C} = L_i L^i$$

GCA Casmir

- The Casmir operators act as,

$$C_2 f_{\Delta,\xi}(x,y) = x^2(1-x)\partial_y^2 f_{\Delta,\xi}(x,y) = \xi^2 f_{\Delta,\xi}(x,y)$$

$$C_4 f_{\Delta,\xi}(x,y) = A^2 f_{\Delta,\xi}(x,y) = \xi^2(\Delta-1)^2 f_{\Delta,\xi}(x,y)$$

  where

$$A = \frac{1}{2}((3x-2)xy\partial_y^2 + 2x^2(x-1)\partial_x\partial_y + 2x^2\partial_x)$$

- These Casmir operators can also be obtained by taking the non-relativistic limit of CFT_2.

Eigenfunction of the Casmir Operators

- Consider the eigenfunctions of C_2 first, and then expand the degenerate eigenfunctions by C_4 .

$$f(x,y) = g(x)e^{\frac{\xi y}{x\sqrt{1-x}}}$$

- A toy model to show how to deal with the ∂_ξ blocks.

$$\sum_{\xi,\alpha} P_{\xi,\alpha} z^\alpha e^{\xi z} = \int_{\#-i\infty}^{\#+i\infty} \sum_{\{\xi_0,\alpha\}} \frac{P_{\xi_0,\alpha}\Gamma[\alpha+1]}{(\xi-\xi_0)^{\alpha+1}} e^{\xi z} d\xi, \quad z>0$$

- Inverse the coefficient by Laplace transformation, which is linear. Close the contour properly to the left side.

Eigenfunction of the Casmir Operators

- For $x > 1$, $\xi \to i\xi$. $z = \frac{y}{\sqrt{|1-x|x}}$.
- Laplace transformation $\to$ Bilateral Laplace transformation. Close the contour separately for each term properly.

$$\frac{G_4}{G_2 G_2} = \int \boxed{Z_\xi(x) e^{\frac{\xi y}{x\sqrt{|1-x|}}} + X_\xi(x) e^{\frac{-\xi y}{x\sqrt{|1-x|}}}} d\xi$$

- It "defines" Z, X point-wise in x.
- For $\xi = 0$ sector, expand it in terms of the eigenfunctions of $\tilde{C}$. [Maldacene, Stanford 16 ';Murugan, Stanford, Witten 17 ']

Eigenfunction of the Casmir Operators

- Expand the boxed terms in terms of the eigenfunctions of $C_4 = A^2$.

$$AZ_\xi(x)e^{\frac{\xi y}{x\sqrt{|1-x|}}} = \xi e^{\frac{\xi y}{x\sqrt{|1-x|}}} DZ_\xi(x)$$

  where

$$D = \frac{2x(x-1)\partial_x + (x-2)}{\sqrt{|1-x|}}$$

- It is impossible to find a measure to make C_2 and A Hermitian simultaneously.
- Instead, it becomes a Strum-Liouville problem, considering A^2, with the measure (in the integral of x),

$$\tilde{\mu}(x) = \frac{\sqrt{|1-x|}}{x^3}$$

- The boundary conditions say that

$$f'(x)|_{x=2} = 0, \quad f \to 0 \textit{ faster than } x$$

Eigenfunction of the Casmir Operators

- There are matching conditions to cancel the potential "boundary" terms in the integral, at $x = 1$. The solution on $[0, 1]$ can be obtained by such matching conditions.
- The general solutions to A^2 are,

$$a_1 A_{\Delta,\xi} + a_2 A_{2-\Delta,-\xi} + a_3 A_{\Delta,-\xi} + a_4 A_{2-\Delta,\xi}$$

where

$$A_{\Delta,\xi} = \frac{x^\Delta (1-\sqrt{1-x})^{2-2\Delta}}{\sqrt{1-x}} e^{\frac{\xi y}{x\sqrt{|1-x|}}}$$

- It is still a solution under the symmetry,

$$\Delta \leftrightarrow 2-\Delta, \quad \xi \leftrightarrow -\xi$$

Symmetry of the 4-pt Function

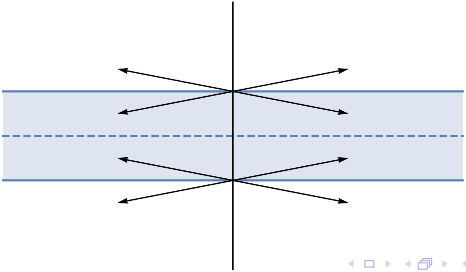
- The 4-pt function is covariant while $\frac{G_4}{G_2 G_2}$ is invariant under the global Galilean conformal transformation.
- Invariant under the exchange of $(1,2) \leftrightarrow (3,4)$(Shadow symmetry).

$$\Delta, \xi \leftrightarrow 2 - \Delta, -\xi$$

- Invariant under the exchange of $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$.

$$x \to \frac{x}{x-1}, \qquad y \to -\frac{1}{(x-1)^2} y$$

- The region required is a strip $x \in [0,2],\ y \in (-\infty, \infty)$.



Boundary conditions

- Boundary conditions:

$$f(x=2,y)=f(x=2,-y)$$

- This is valid for arbitrary Δ, ξ, which gives the constraints on the coefficients

$$a_1 = a_3, \quad a_2 = a_4$$

- $y=0, x=2$ is a fixed point, which is consistent with the boundary conditions to make C_4 Hermitian.

$$\partial_x f(x=2, y=0) = 0$$

- This gives the solution on $x \in [1, \infty)$.

Eigenfunctions

- To make the eigenvalues real,

$$\begin{cases} \xi = & ir \\ \Delta = & 1 + is \end{cases} \quad or \quad \begin{cases} \xi = & ir \\ \Delta = & s \end{cases}$$

  where r and s are real.

- The solution on $x \in [1, \infty)$ is

$$e^{2i\sqrt{(\Delta-1)^2} ArcTan\sqrt{x-1}} \frac{x}{\sqrt{x-1}} + be^{-2i\sqrt{(\Delta-1)^2} ArcTan\sqrt{x-1}} \frac{x}{\sqrt{x-1}}$$

  with

$$b = e^{i\pi\sqrt{(\Delta-1)^2}}$$

Eigenfunctions

- Consider the $e^{\frac{\xi y}{\sqrt{|1-x|x}}}$ part, one gets $a_1 = a_3$, $a_2 = a_4$ for $x \in [0, 1]$. We consider the $y = 0$ slice in the following discussion for simplicity.
- From the matching condition at $x = 1$,

$$f(x \to 1^+) = A + \frac{B}{\sqrt{x-1}}, \qquad f(x \to 1^-) = A + \frac{B}{\sqrt{1-x}}$$

One gets the solution on $x \in [0, 1]$,

$$f(x, y) = \frac{1}{4}(a_1(A_{\Delta,\xi} + A_{\Delta,-\xi}) + a_2(A_{2-\Delta,-\xi} + A_{2-\Delta,\xi})) \quad (1)$$

where

$$a_1 = \frac{\Delta + e^{i\pi\sqrt{(\Delta-1)^2}}\left(\Delta - i\sqrt{(\Delta-1)^2} - 1\right) + i\sqrt{(\Delta-1)^2} - 1}{2(\Delta - 1)}$$

$$a_2 = \frac{\Delta + e^{i\pi\sqrt{(\Delta-1)^2}}\left(\Delta + i\sqrt{(\Delta-1)^2} - 1\right) - i\sqrt{(\Delta-1)^2} - 1}{2(\Delta - 1)}$$

Eigenfunctions

- Consider the $x \to 0$ behaviour,

$$A_{\Delta,\xi} \sim x^{2-\Delta}$$

- The fall off condition at $x = 0$ says,

$$f(x) \sim x^a, \quad 2\mathrm{Re}\, a - 3 \geq -1$$

- $\Delta = 1 + is$ is marginally allowable.
- a_1, a_2 cannot vanish for certain real Δ, so there is no real $\Delta \neq 1$ eigenfunctions.

Eigenfunctions

- The eigenfunctions are piece-wise functions with the symmetry $f(x,y) = f(\frac{x}{x-1}, -\frac{1}{(x-1)^2}y)$.

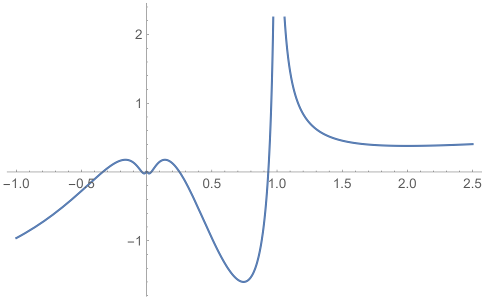


Figure: $s = \frac{3}{2}$, at $y = 0$ slice

- Oscillatory for $x < 1$, monotonic for $x > 1$ at $y = 0$ slice (real).
- At fixed x slice, they are periodic in y (with non-vanishing imaginary part).

Orthogonality and Completeness

- The total measure is,

$$\mu(x,y) = \frac{1}{x\sqrt{|1-x|}}\frac{\sqrt{|1-x|}}{x^3} = \frac{1}{x^4}$$

- Consider $\Delta = 1 + is$, with positive s. The singular part of the inner product which contributes to the delta function is the integral over the small x region.

$$(\psi_{\Delta,\xi}, \psi_{\Delta\,',\xi\,'}) \sim (1+e^{-2\pi s})2\pi\delta(s-s\,')\delta(r-r\,')$$

- The completeness relation is than,(the $\xi = 0$ sector should be replaced)

$$\int \psi_{\Delta,\xi}(x,y)\psi_{\Delta,\xi}(x\,',y\,')d\Delta d\xi = Nx^4\delta(x-x\,')\Delta(y-y\,')$$

- One can check the completeness relation by using the identity,

$$\psi_{\Delta,\xi}(x\,',y\,') = \int \delta(x-x\,')\delta(y-y\,')\psi_{\Delta,\xi}(x,y)dxdy$$

Partial Wave Expansion

- The 4-pt function admits a partial wave expansion, with $\Delta = 1 + is,\ \xi = ir$

$$\frac{G_4}{G_2 G_2} = \frac{1}{2\pi i}\int P(\Delta,\xi)\psi_{\Delta,\xi} d\Delta d\xi$$

- The "Euclidean-like" inversion formula reads,

$$\boxed{P(\Delta,\xi) = (\frac{G_4}{G_2 G_2}, \psi_{\Delta,\xi})}$$

$$\boxed{P(\Delta,\xi=0) = (\frac{G_4}{G_2 G_2}, \tilde{\psi}_{\Delta,\xi=0})\ '}$$

- The physical information can be obtained by,

$$P_{\Delta_0,\xi_0,\alpha} = \frac{1}{\Gamma[\alpha+1]} Res|_{\Delta=\Delta_0,\xi=\xi_0}[(\xi-\xi_0)^\alpha P(\Delta,\xi)]$$

Lorentzian Inversion formula

There are two main purposes of studying the LIF.

- To study the Regge limit, the analytic structure make it possible to re-sum the divergent conformal blocks to something making sense by Sommerfeld-Watson trick.
- Study the analytic bootstrap.

The two purposes are unified in one Lorentzian inversion formula in higher dimensional CFTs. It is not the case in CFT_1 [Simmons-Duffin, Stanford, Witten 17 '] and Galilean conformal field theories.

Lorentzian Inversion formula

How to re-sum the blocks to proper Regge limit?

- The contributions to the 4-pt function from the partial wave expansion come from both the continues spectrum and discrete spectrum. The continues part is still analytic in Δ, ξ. Summing the discrete part (only appear in the eigenfunctions of $Sl(2,R)$ Casmir) directly leads to divergence.[Simmons-Duffin, Stanford, Witten 17 ']

$$I_n = \int_{-\infty}^{\infty} \frac{dx}{x^2} g(x) \psi_n{}'(x)$$

- The behaviour at infinity is controlled by Regge limit. The "Lorentzian-like" inversion formula can be obtained by contour deformation and analytic continuation of the blocks.

$$\tilde{I}_{\Delta,J} = \frac{\Gamma(n)^2}{\Gamma(2n)} ((-)^J \int_{\infty}^{0} \frac{dx}{x^2} \hat{k}_{2\Delta}(x) dDisc[g] + \int_0^1 \frac{dx}{x^2} k_{2\Delta}(x) dDisc[g(x)])$$

- $I_n = \tilde{I}_n$ only for integer n. $\tilde{I}$ is used in the Sommerfeld-Watson trick to give the correct Regge behaviour.

Lorentzian Inversion formula

- The Euclidean and Lorentz conformal group are different. For CFT_1 and GCA, there is no such difference.
- The above proposal of the discrete part cannot give any information of the "physical spectrum".

$$I_\Delta \neq \tilde{I}_\Delta$$

- It is crucial to express the $P(\Delta, \xi)$ as the integral transformation of the $dDisc[\frac{G_4}{G_2 G_2}]$.
- The integral kernel is evaluated by deforming the contour of the "Euclidean-like" inversion formula.

$$\boxed{P(\Delta, \xi) = \int K(x, y) dDisc[\frac{G_4}{G_2 G_2}] \mu(x) dx dy}$$

Loretzian-like Inversion Formula

Lorentzian Inversion formula

- The t-channel information is contained in the $dDisc[\frac{G_4}{G_2G_2}]$, while the s-channel information is contained in $P(\Delta, \xi)$. The "Lorentzian-like" inversion formula itself is the crossing equation!
- The $\xi = 0$ sector (CFT_1 case, [Mazac 18 ']) and $\xi \neq 0$ sector should be dealt with separately.
- The kernel is not the eigenfunctions of the s-channel Casmir operators, which is dependent on the Δ_O, ξ_O of the external operators. There is no closed form of the kernel for general Δ_O, ξ_O.
- The kernel $K(x, y)$ satisfies a list of functional equations to make the two ways calculating the same coefficients. There is no closed form for such kernel.
- We are studying the large Δ limit of such kernel to do the bootstrap.

Conclusion and Discussion

- Considering the theories with Lifshitz symmetry, the global symmetries are enhanced to infinite many local symmetries under certain assumptions.
- One of them is Galilean conformal field theories. We calculate its Kac determinant, and find there is no nontrivial minimal models.
- We examine the block expansion of the 4-pt functions in GCA, and expand it in terms of partial waves, give a Eucliden-like inversion formula.
- Hopefully, in the large Δ limit, one may get universal information of the coefficient.

Thanks for Your Attention!

2D Lifshitz Scaling

- The commutation relations are as follows,

$$[H, \bar{H}] = 0, [D, H] = -aH, [D, \bar{H}] = -b\bar{H}$$

$$[B, H] = -\bar{H}, [B, \bar{H}] = 0, [B, D] = (b - a)B$$

- The charges and conservation laws are[Hofman, Rollier, 14 '],

$$Q = \int \star J, \quad \star = H_{\mu\nu}, \qquad \nabla_\mu J^\mu = 0$$

where $H_{\mu\nu}$ serves as the volume in the Newton-Cartan Geometry.

Geometry

- Such theories should be defined on the Newton-Carton Geometry, e.g. WCFT.[Hofman, Rollier, 14 ']
- A contravariant tensor $\gamma = \gamma^{\mu\nu}\partial_\mu \otimes \partial_\nu$, a orthogonal time 1-form $\tau = \tau_\mu dx^\mu$. The metric γ has one positive eigenvalue and one vanishing eigenvalue. Non-dynamical satial metric on slices orthogonal to τ.
- A fibre bundle with base manifold (time), and spatial slices as fibres.
- Dynamical connection, compatible with γ, τ, and also a scaling structure,

$$J = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

- For $a = b$, it is the same as $R^{\mu\nu}_{\lambda\sigma} = R^{\nu\mu}_{\sigma\lambda}$.[Bagchi, Gopakumar, 14 ']

GCA Casmir from Taking Limit

- The quadratic and quartic Casmir operators in CFT_2 act as,

$$C2f_{\Delta,J}(z,\bar{z}) = (D_z + D_{\bar{z}})f_{\Delta,J}(z,\bar{z}) = \frac{1}{2}[J^2 + \Delta(\Delta-2)]f_{\Delta,J}(z,\bar{z})$$

$$C4f_{\Delta,J}(z,\bar{z}) = (D_z - D_{\bar{z}})^2 f_{\Delta,J}(z,\bar{z}) = J^2(\Delta-1)^2 f_{\Delta,J}(z,\bar{z})$$

  where

$$D_z = z^2(1-z)\partial_z^2 - z^2\partial_z, \quad D_{\bar{z}} = \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}$$

- After taking the $\epsilon \to 0$ non-relativistic limit, one gets the eigen equation in the Galilean conformal theory.