SPIN AND HIGHER MULTIPOLE CORRECTIONS TO EMRIS

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CONTENTS

INTRODUCTION & MOTIVATION

- GR has passed varies tests, including deflection of light, precession of Mercury…
- Recently, gravitational waves, one of its prediction, has been detected by LIGO.
- Binary black hole (BBH) systems are perfect laboratory to test GR.

1) Newton gravity:

   i) two-body problem, exactly solvable

   ii) three-body problem?

2) Einstein gravity:

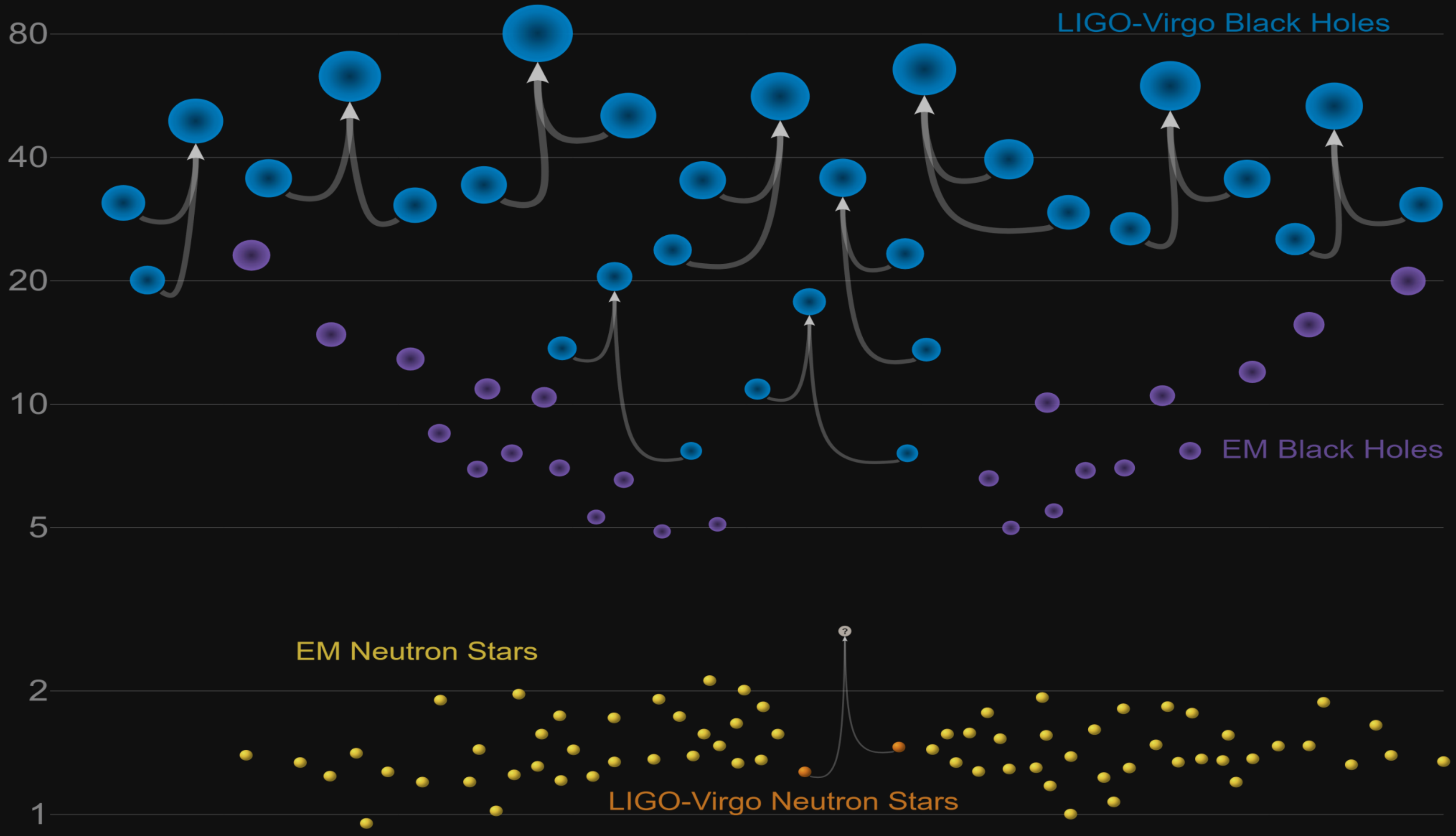
   i) one-body problem, Schwarzchild, Kerr

   ii) two-body problem, not easy to find an analytic solution

3) Gravitational waves are radiated from BBHs.

Masses in the Stellar Graveyard

in Solar Masses



LIGO-Virgo | Frank Elavsky | Northwestern

INTRODUCTION & MOTIVATION

- Two facts

  1) the mass ratio of two BHs (NS): $\frac{M_1}{M_2} \approx 1$

  2) The mass of the BH in this picture: $\mathrm{M} \sim 10 M_{solar}$, stellar black holes
- BHs:

  1) Stellar black hole: gravitational collapse of a star , $1 \sim 10^2 M_{solar}$

  2) Intermediate mass black hole (IMBH): no strong evidence, $10^2 \sim 10^5 M_{solar}$

  3) Supermassive black hole (SMBH): center of galaxies, $10^5 \sim 10^9 M_{solar}$

INTRODUCTION & MOTIVATION

- The parameter space of BBH: $m, S, M, J, l_{orb}, \ldots$
- LIGO just tests the region: $\frac{m}{M} \approx 1,\ M \approx 10M_{solar}$
- Intermediate and extreme mass ratio: $\mathrm{q} \equiv \frac{m}{M} \ll 1$
- The small BH $m \approx M_{solar}$
- Intermediate mass ratio $q \approx (10^{-2} \sim 10^{-5})$
- Extreme mass ratio $q \approx (10^{-6} \sim 10^{-9})$
- We will discuss the perturbation theory to compute gravitational wave in the region $q \ll 1$

ASSUMPTIONS

- Large BH
  1) Near-extreme Kerr black hole (High spin Kerr black hole)
  2) Near horizon region: emergence of conformal symmetry

     the last stage of black hole merger

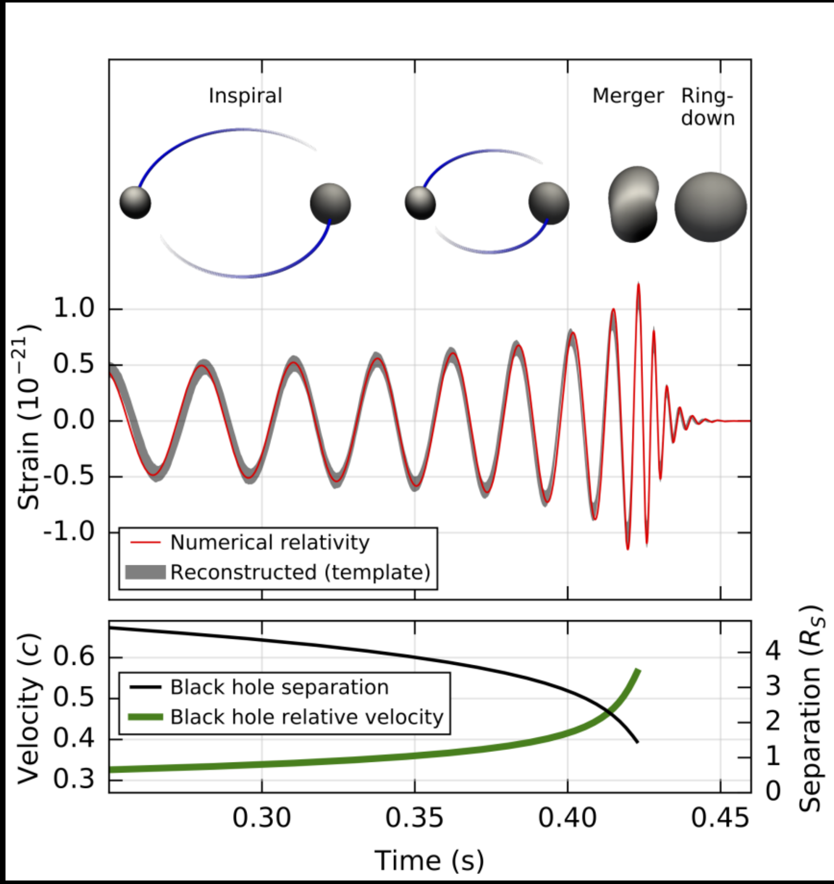
- $\lambda = \sqrt{1 - \frac{J^2}{M^4}} \ll 1$

  J.Bardeen &G.Horowitz (1999)

  M.Guica, T.Hartman, W.Song &A.Strominger (2009)

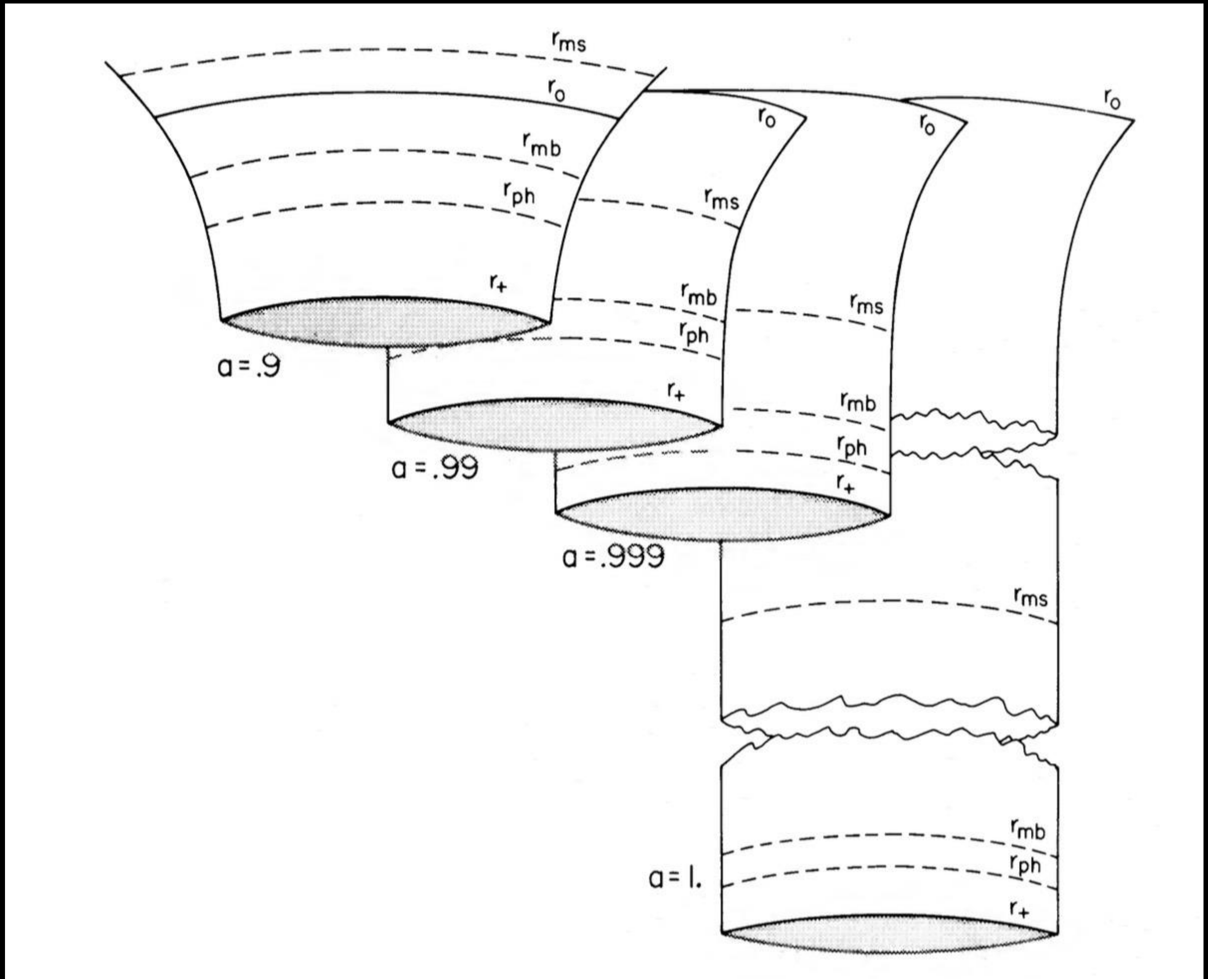
ASSUMPTIONS

- Coalescence of a binary black hole
- Three steps(phases):
  1) Inspiral
  2) Merger
  3) Ringdown
- Large high spin Kerr black hole as a background
- Three patches of Kerr black hole
  1) far region
  2) NHEK region
  3) near-NHEK region



KERR BLACK HOLE

- Three patches of a high spin Kerr black hole



Last stage of a small black hole
Falls into a large high spin Kerr black hole
is in NHEK and near-NHEK region

- J.Bardeen, W.Press, S.Teukolsky (1972)

FAR REGION

- Far region

$$ds^2 = -(1-\frac{2M\hat{r}}{\Sigma})d\hat{t}^2 + \frac{\Sigma}{\Delta}d\hat{r}^2 + \Sigma d\theta^2 + (\hat{r}^2 + a^2 + \frac{2Ma^2\hat{r}\sin^2\theta}{\Sigma})\sin^2\theta d\hat{\phi}^2 - \frac{4Ma\hat{r}\sin^2\theta}{\Sigma}d\hat{t}d\hat{\phi}$$

where

$$\Delta \equiv \hat{r}^2 - 2M\hat{r} + a^2, \qquad \Sigma \equiv \hat{r}^2 + a^2\cos^2\theta.$$

$$\hat{x} \equiv \frac{\hat{r} - \hat{r}_+}{\hat{r}_+}$$

- $\hat{x} \ll 1, near\ horizon\ region$
- $\hat{x} \to \infty$, observer

NHEK REGION

- NHEK region

$$ds^2 = 2M^2\Gamma(\theta)\left(-R^2dT^2 + \frac{dR^2}{R^2} + d\theta^2 + \Lambda^2(\theta)(d\Phi + RdT)^2\right)$$

where

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}$$

- It can be obtained by coordinate transformation and take the limit $\lambda \to 0$

$$\begin{aligned} T &= \frac{\hat{t}}{2M}\lambda^{2/3}, \\ R &= \frac{\hat{r}-\hat{r}_+}{M}\lambda^{-2/3}, \\ \Phi &= \hat{\phi} - \Omega_{ext}\hat{t}, \qquad \Omega_{ext} \equiv \frac{1}{2M} \end{aligned}$$

NEAR-NHEK REGION

- Near-NHEK region

$$ds^2 = 2M^2\Gamma(\theta)\left(-r(r+2\kappa)dt^2 + \frac{dr^2}{r(r+2\kappa)} + d\theta^2 + \Lambda^2(\theta)(d\phi + (r+\kappa)dt)^2\right)$$

It can be obtained by coordinate transformation and take the limit $\lambda \to 0$

$$t = \frac{\hat{t}}{2M\kappa}\lambda,$$
$$r = \kappa\frac{\hat{r} - \hat{r}_+}{M\lambda},$$
$$\phi = \hat{\phi} - \frac{\hat{t}}{2M},$$

$r \to 0, horizon$

$r \to \infty, attach\ to\ far\ region$

ASSUMPTIONS

- Small black hole
- 1) mass m (p^μ)
- 2) spin S ($S^{\rho\sigma}$)
- 3) black hole is not a point particle, it has a size!
- 4) As a first step, we ignore any backreaction from gravitational waves
- How to describe the movement of an extended object in curved spacetime?
- Generalization of geodesics

MATHISSON-PAPAPETROU-DIXON FORMALISM

- Geodesics of a point particle without spin $u^\mu \nabla_\mu u^\rho = 0$
- A particle with momentum p and spin S (MP equation)
- $p^\mu \equiv \int T^{\mu\rho} d\Sigma_\rho, \; S^{\alpha\beta} \equiv \int (x^\alpha - z^\alpha) T^{\beta\gamma} d\Sigma_\gamma - (\alpha \leftrightarrow \beta)$
- Conservation of stress tensor

$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta}$$

$$\frac{DS^{\mu\nu}}{D\tau} = p^\mu u^\nu - p^\nu u^\mu$$

- Spin Supplementary Condition (SSC)

$$S^{\mu\nu} p_\nu = 0$$

MATHISSON-PAPAPETROU-DIXON FORMALISM

- Evolution equations of an extended body
- Force and torque: presence of higher multipoles
- $2^N - pole$: described by a tensor with N+2 indices
- $J_{\mu_1 \mu_N \alpha\beta\gamma\delta}$ with symmetry structure

$$J^{\mu_1 \mu_{N-2}\alpha\beta\gamma\delta} = J^{(\mu_1 \mu_{N-2})[\alpha\beta][\gamma\delta]},$$
$$J^{\mu_1 \mu_{N-2}\alpha[\beta\gamma\delta]} = 0,$$
$$J^{\mu_1 \mu_{N-3}[\mu_{N-2}\alpha\beta]\gamma\delta} = 0, \quad \text{for} \quad N \geq 3$$
$$n_{\mu_1} J^{\mu_1 \mu_{N-2}\alpha\beta\gamma\delta} = 0, \quad \text{for} \quad N \geq 3.$$

- $g_{\alpha\beta,\mu_1\cdots\mu_N}$: a extension of metric in the sense of Veblen and Thomas

$$\frac{Dp^\mu}{D\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} u^\nu S^{\alpha\beta} + \mathcal{F}^\mu$$
$$\frac{DS^{\mu\nu}}{D\tau} = p^\mu u^\nu - p^\nu u^\mu + \mathcal{L}^{\mu\nu}$$

$$\mathcal{F}^\mu = \frac{1}{2} \sum_{N\geq 2} \frac{1}{N!} m^{\mu_1\cdots\mu_N\lambda\kappa} \nabla^\mu g_{\lambda\kappa,\mu_1\cdots\mu_N},$$
$$\mathcal{L}^{\mu\nu} = \sum_{N\geq 2} \frac{1}{(N-1)!} g^{\rho[\mu} m^{\nu]\mu_1\cdots\mu_{N-1}\alpha\beta} g_{\{\rho\mu,\nu\}\mu_1\cdots\mu_{N-1}}$$

$$m^{\mu_1\cdots\mu_N\rho\sigma\kappa} = \frac{4N}{N+2} J^{(\mu_1\cdots\mu_N|\sigma|\rho)\kappa}$$

$$g_{\{\alpha\beta,\gamma\}\delta\cdots} = g_{\alpha\beta,\gamma\delta\cdots} - g_{\beta\gamma,\alpha\delta\cdots} + g_{\gamma\alpha,\beta\delta\cdots}$$

MATHISSON-PAPAPETROU-DIXON FORMALISM

- Mass:
  1) $\underline{m}^2 = -p^2$
  2) $m = -p \cdot u$
- In general, $\underline{m} \neq m$
- Spin:

$$S_\mu = \frac{1}{2\underline{m}} \epsilon_{\mu\alpha\beta\gamma} p^\alpha S^{\beta\gamma}$$

- Spin length:

$$S^2 = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta} = S^\mu S_\mu$$

- $\underline{m}, m, S$ *are not conserved in the presence of higher multipoles*

MATHISSON-PAPAPETROU-DIXON FORMALISM

- Conserved quantities: Given a Killing vector ξ^α

  $Q_\xi = \xi_\alpha p^\alpha + \frac{1}{2} S^{\alpha\beta} \nabla_\alpha \xi_\beta$ is conserved even in the presence of multipoles
- Stress tensor: MPD equations are equivalent to $\nabla_\mu T^{\mu\rho} = 0$
- Up to quadrupole,

$$\mathcal{F}^\mu = -\frac{1}{6} J^{\alpha\beta\gamma\delta} \nabla^\mu R_{\alpha\beta\gamma\delta},$$

$$\mathcal{L}^{\mu\nu} = \frac{4}{3} J^{\alpha\beta\gamma[\mu} R^{\nu]}{}_{\gamma\alpha\beta}.$$

- The stress tensor is

$$T^{\mu\nu} = \int d\tau [(p^{(\mu} u^{\nu)}) \mathcal{D} + \frac{1}{3} R_{\alpha\beta\gamma}{}^{(\mu} J^{\nu)\gamma\beta\alpha} \mathcal{D} - \nabla_\alpha (S^{\alpha(\mu} u^{\nu)} \mathcal{D}) - \frac{2}{3} \nabla_\alpha \nabla_\beta (J^{\alpha(\mu\nu)\beta} \mathcal{D})]$$

$$\mathcal{D} = \frac{1}{\sqrt{-g}} \delta^{(4)}(x^\mu - x_*^\mu(\tau))$$

QUADRUPOLE MODEL

- To solve MPD equations, one should construct explicit higher multipole model
- Some effects that could contribute to quadrupole

  1) spin-induced quadrupole $S^{\alpha[\mu}p^{\nu]}S_{\alpha}{}^{[\rho}p^{\sigma]}$

  2)gravito-electric tidal field induced quadrupole $p^{[\mu}E^{\nu][\rho}p^{\sigma]}$

  3)gravito-magnetic tidal field induced quadrupole $(p^{[\mu}Q^{\nu]\rho\sigma}+p^{[\sigma}Q^{\rho]\nu\mu})]$
- The quadrupole is a linear combination of these terms

$$J^{\mu\nu\rho\sigma} = \frac{m}{\mathrm{m}^3}[\frac{3\kappa s^2}{\mathrm{m}}S^{\alpha[\mu}p^{\nu]}S_{\alpha}{}^{[\rho}p^{\sigma]} + 3\mu_2 p^{[\mu}E^{\nu][\rho}p^{\sigma]} + 2\sigma_2(p^{[\mu}Q^{\nu]\rho\sigma}+p^{[\sigma}Q^{\rho]\nu\mu})]$$

- J.Steinhoff & D.Puetzfeld (2012)

$$\begin{aligned} Q^{\mu\nu\rho} &= \epsilon^{\rho\nu}{}_{\alpha\beta}p^{\alpha}B^{\mu\beta}, \\ E_{\mu\nu} &= \frac{1}{\mathrm{m}^2}R_{\mu\rho\nu\sigma}p^{\rho}p^{\sigma}, \\ B_{\mu\nu} &= \frac{1}{2\mathrm{m}^2}\epsilon_{\mu\alpha\beta\gamma}R_{\nu\delta}{}^{\beta\gamma}p^{\alpha}p^{\delta} \end{aligned}$$

QUADRUPOLE MODEL

- Dimensional analysis

$$[x^\mu] = -1,\ [\tau] = -1,\ [p^\mu] = 1,\ [u^\mu] = 0,\ [S^{\mu\nu}] = 0,\ [g_{\mu\nu}] = 0,$$
$$[R^\mu{}_{\nu\rho\sigma}] = 2,\ [J^{\mu\nu\rho\sigma}] = -1, [G_N] = -2,\ [c] = 0.$$
$$[\kappa_{S^2}] = 0,\ [\mu_2] = -3,\ [\sigma_2] = -3.$$

- $\kappa_{S^2} = 1$ for black hole, $\kappa_{S^2} \approx 5$ for neutron stars

  W.Laarakkers &E.Poisson (1999)
- $\underline{m}, m$ *are non − conserved, though they are equal up to* $O(S^3)$
- μ is conserved up to $O(S^3)$, it is the mass term in perturbation theory

$$\mu \;=\; m + \frac{\kappa_{S^2}}{2m} E_{\mu\nu} S^\mu{}_\alpha S^{\alpha\nu} + \frac{\mu_2}{4} E_{\mu\nu} E^{\mu\nu} + \frac{2}{3} \sigma_2 B_{\mu\nu} B^{\mu\nu}$$

CIRCULAR ORBIT

- Solve MPD equations in near-NHEK region to find the trajectory of the small BH
- Spinless case:

  Equatorial plane

$$r = r_0 = \frac{2\kappa\ell}{\sqrt{3(\ell^2 - \ell_*^2)}} - \kappa,$$
$$\phi = \phi_0 - \frac{3}{4}(r_0 + \kappa)t.$$

$$e = -\frac{\sqrt{3}\kappa}{2}\sqrt{\ell^2 - \ell_*^2}$$

$$q \equiv \frac{\mu}{M} \ll 1.$$

- Spin and size effect: small mass ratio expansion

$$\chi \equiv \frac{S}{\mu^2}$$

$$-1 \leq \chi \leq 1$$

- In small q expansion, one can prove

$$p^\mu = O(q^1), S^{\alpha\beta} = O(q^2), \mu_2 = O(q^5), \sigma_2 = O(q^5)$$

- Gravito-electric and magnetic tidal deformations are higher order

CIRCULAR ORBIT

- Assumptions:

$$r = r_0, \qquad \theta = \frac{\pi}{2}, \qquad \phi = -\alpha r_0 t$$

$$\kappa_0 \equiv \frac{\kappa}{r_0}$$

$$u^t = \frac{1}{M r_0 \sqrt{8(1+\kappa_0)\alpha - (3 + 4\alpha^2 + 6\kappa_0 + 4\kappa_0^2)}}, \qquad u^\phi = -\alpha r_0 u^t,$$

$$S^{tr} = \frac{(1+\kappa_0)\chi q^2}{\lambda_0}\left(1 + \frac{6(1+2\kappa_0)\chi q}{\lambda_0^2} + \mathcal{O}(q^2)\right),$$

$$S^{r\phi} = \frac{r_0\kappa_0^2\chi q^2}{\lambda_0}\left(1 + \frac{9(1+\kappa_0)^2(1+2\kappa_0)\chi q}{2\kappa_0^2\lambda_0^2} + \mathcal{O}(q^2)\right),$$

$$p^t = \frac{2q}{r_0\lambda_0}\left(1 + \frac{3(1+\kappa_0)^2\chi q}{2\lambda_0^2} + \frac{(3(1+\kappa_0)^2(6+12\kappa_0+\kappa_0^2) + 2(-9+\kappa_0(-36-36\kappa_0+\kappa_0^3))\kappa_{S^2})\chi^2 q^2}{2\lambda_0^4} + \mathcal{O}(q^3)\right),$$

$$p^\phi = -\frac{3(1+\kappa_0)q}{2\lambda_0}\left(1 + \frac{2\kappa_0^2\chi q}{\lambda_0^2} + \frac{(2\kappa_{S^2}(-9-36\kappa_0-36\kappa_0^2+\kappa_0^4) + 9 + 36\kappa_0 + 57\kappa_0^2 + 42\kappa_0^3 + 4\kappa_0^4)\chi^2 q^2}{2\lambda_0^4} + \mathcal{O}(q^3)\right),$$

CIRCULAR ORBIT

- Solution

$$\alpha = \frac{3}{4}(1+\kappa_0)(1-\frac{\chi q}{2}+\frac{1}{4}(4\kappa_{S^2}-5)\chi^2 q^2+\mathcal{O}(q^3))$$

$$e \equiv \frac{Q_{-\partial_t}}{\mu} = -\frac{2Mr_0\kappa_0^2}{\lambda_0}(1+\frac{(9+18\kappa_0+\kappa_0^2)\chi q}{2\lambda_0^2} + \frac{(27(1+\kappa_0)^2(1+2\kappa_0)+2(-9+\kappa_0(-36-36\kappa_0+\kappa_0^3))\kappa_{S^2})\chi^2 q^2}{2\lambda_0^4}+\mathcal{O}(q^3))$$

$$\ell \equiv \frac{Q_{\partial_\phi}}{\mu} = \frac{2M(1+\kappa_0)}{\lambda_0}(1+\frac{(3+\kappa_0(6+\kappa_0))\chi q}{\lambda_0^2} + \frac{(2\kappa_{S^2}(-9-36\kappa_0-36\kappa_0^2+\kappa_0^4)+9+36\kappa_0+69\kappa_0^2+66\kappa_0^3)\chi^2 q^2}{2\lambda_0^4}+\mathcal{O}(q^3)),$$

$$\underline{\mathrm{m}} = Mq(1-\frac{(3+6\kappa_0+\kappa_0^2)\kappa_{S^2}\chi^2 q^2}{\lambda_0^2}+\mathcal{O}(q^3)).$$

$$\lambda_0 = \sqrt{3+6\kappa_0-\kappa_0^2}$$

NHEK: $\kappa_0 \to 0$

CIRCULAR ORBIT

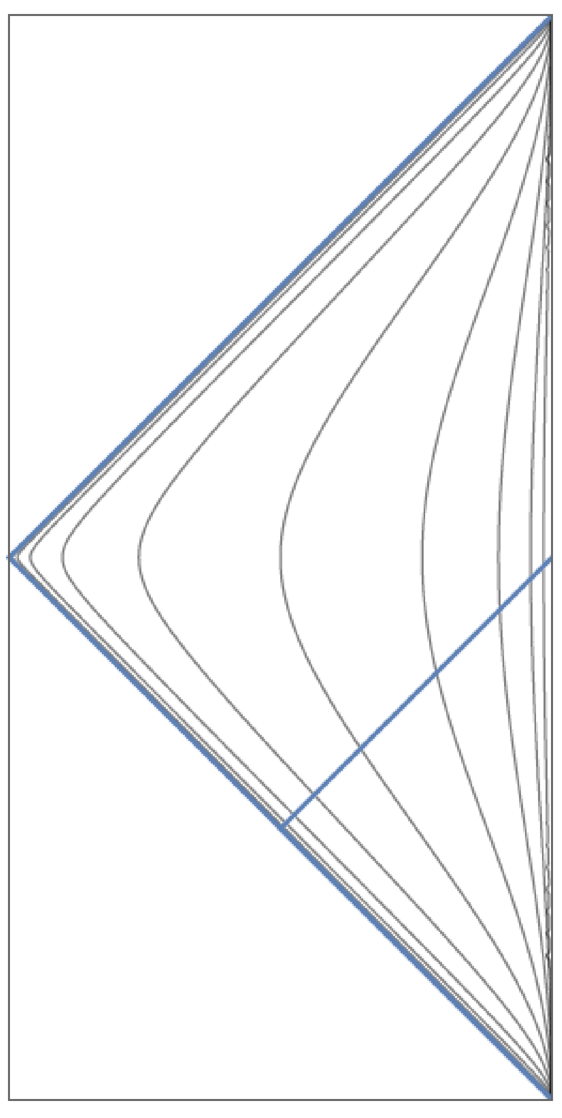
$$
\begin{aligned}
S^{tr} &= \frac{\ell \chi q^2}{\sqrt{3}\ell_*}(1+2\chi q) + O(q^4), \\
S^{r\phi} &= -\frac{e\chi q^2}{\sqrt{3}\ell_*}\Big(1+\frac{2\chi q}{1-\frac{\ell_*^2}{\ell^2}}\Big) + O(q^4), \\
p^t &= -\frac{\sqrt{3}\ell_* q}{2e}\Big(\frac{\ell^2}{\ell_*^2}-1\Big)\Big(1-\frac{\chi^2 q^2}{2}\Big) + O(q^4), \\
p^\phi &= -\frac{\sqrt{3}\ell q}{2\ell_*}\Big(1+\big(\frac{1}{2}-\kappa_{S^2}\big)\chi^2 q^2\Big) + O(q^4), \\
\underline{\mathrm{m}} &= Mq\Big(1-\frac{\kappa_{S^2}}{2}\big(\frac{\ell^2}{\ell_*^2}+1\big)\chi^2 q^2\Big) + O(q^4), \\
\frac{\alpha}{\kappa_0} &= \frac{\sqrt{3}\ell}{2\sqrt{\ell^2-\ell_*^2}}\Big(1+\frac{1}{2}(\kappa_{S^2}-1)\chi^2 q^2\Big) + O(q^3).
\end{aligned}
$$

$$
\ell_*[\chi q] \equiv \frac{2M}{\sqrt{3}}\Big(1+\chi q+\big(\frac{1}{2}-\kappa_{S^2}\big)(\chi q)^2\Big) + O(q^3).
$$

- l_* is the orbital angular momentum of NHEK circular orbit, critical angular momentum in near-NHEK

GENERAL EQUATORIAL ORBITS

- Conformal transformation: $SL(2,R) \times U(1) \times PT$
  1) preserve NHEK
  2) preserve near-NHEK
  3) NHEK↔near-NHEK
- Near-NHEK: Circular(l_*)
  NHEK: $Circular_*$
- Spinless case: all plunging or osculating equatorial orbits entering into near-NHEK or NHEK are conformally related to a circular orbit.
  G.Compere, K.Fransen, T.Hertog, J.Long (2017)
- MPD equations are covariant. We expect any equatorial orbit can be obtained by applying conformal maps.



GRAVITATIONAL WAVES

- Teukolsky equation, Linearized perturbation equation of Kerr black hole

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

Newman-Penrose formalism

- two null vectors l^μ, n^μ and one complex null vector m^μ, $l \cdot n = -m \cdot \bar{m} = -1$ and $g_{\mu\nu} = -l_{(\mu} n_{\nu)} + m_{(\mu} \bar{m}_{\nu)}$
- four derivatives

$$D = l^\mu \partial_\mu, \quad \Delta = n^\mu \partial_\mu, \quad \delta = m^\mu \partial_\mu, \quad \bar{\delta} = \bar{m}^\mu \partial_\mu.$$

- twelve spin coefficients
- five Weyl scalars

GRAVITATIONAL WAVES

- Spin coefficicent & Weyl scalar

$$\begin{aligned}
\kappa &= -m^\mu l^\nu \nabla_\nu l_\mu & \sigma &= -m^\mu m^\nu \nabla_\nu l_\mu \\
\lambda &= -n^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu & \nu &= -n^\mu n^\nu \nabla_\nu \bar{m}_\mu \\
\rho &= -m^\mu \bar{m}^\nu \nabla_\nu l_\mu & \mu &= -n^\mu m^\nu \nabla_\nu \bar{m}_\mu \\
\tau &= -m^\mu n^\nu \nabla_\nu l_\mu & \varpi &= -n^\mu l^\nu \nabla_\nu \bar{m}_\mu \\
\epsilon &= -\frac{1}{2}(n^\mu l^\nu \nabla_\nu l_\mu + m^\mu l^\nu \nabla_\nu \bar{m}_\mu) \\
\gamma &= -\frac{1}{2}(n^\mu n^\nu \nabla_\nu l_\mu + m^\mu n^\nu \nabla_\nu \bar{m}_\mu) \\
\alpha &= -\frac{1}{2}(n^\mu \bar{m}^\nu \nabla_\nu l_\mu + m^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu) \\
\beta &= -\frac{1}{2}(n^\mu m^\nu \nabla_\nu l_\mu + m^\mu m^\nu \nabla_\nu \bar{m}_\mu)
\end{aligned}$$

$$\begin{aligned}
\psi_0 &= C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu \\
\psi_1 &= C_{\alpha\beta\mu\nu} l^\alpha n^\beta l^\mu m^\nu \\
\psi_2 &= C_{\alpha\beta\mu\nu} l^\alpha m^\beta \bar{m}^\mu n^\nu \\
\psi_3 &= C_{\alpha\beta\mu\nu} l^\alpha n^\beta \bar{m}^\mu n^\nu \\
\psi_4 &= C_{\alpha\beta\mu\nu} n^\alpha \bar{m}^\beta n^\mu \bar{m}^\nu
\end{aligned}$$

- $\delta\psi_{-2} = \rho^{-4}\delta\psi_4$ encodes complete information of gravitational waves

$$\delta\psi_4(\mathrm{r} \to \infty) = \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times)(r \to \infty)$$

GRAVITATIONAL WAVES

- Gravitational perturbations around Kerr BHs

$$[(D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})(\Delta - 4\gamma + \mu) - (\delta + \overline{\varpi} - \bar{\alpha} - 3\beta - 4\tau)(\bar{\delta} + \varpi - 4\alpha) - 3\psi_2]\delta\psi_0 = 4\pi T_0,$$

$$[(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(D + 4\epsilon - \rho) - (\bar{\delta} - \bar{\tau} + \bar{\beta} + 3\alpha + 4\varpi)(\delta - \tau + 4\beta) - 3\psi_2]\delta\psi_4 = 4\pi T_4$$

- source term: stress tensor $T_{lm} = T_{\mu\nu} l^{\mu} n^{\nu}$.

$$T_0 = (\delta + \overline{\varpi} - \bar{\alpha} - 3\beta - 4\tau)[(D - 2\epsilon - 2\bar{\rho})T_{lm} - (\delta + \overline{\varpi} - 2\bar{\alpha} - 2\beta)T_{ll}] + (D - 3\epsilon + \bar{\epsilon} - 4\rho - \bar{\rho})[(\delta + 2\overline{\varpi} - 2\beta)T_{lm} - (D - 2\epsilon + 2\bar{\epsilon} - \bar{\rho})T_{mm}],$$

$$T_4 = (\Delta - \bar{\gamma} + \bar{\mu} + 3\gamma + 4\mu)[(\bar{\delta} - 2\bar{\tau} + 2\alpha)T_{n\bar{m}} - (\Delta + \bar{\mu} - 2\bar{\gamma} + 2\gamma)T_{\bar{m}\bar{m}}] + (\bar{\delta} + 3\alpha + \bar{\beta} + 4\varpi - \bar{\tau})[(\Delta + 2\bar{\mu} + 2\gamma)T_{n\bar{m}} - (\bar{\delta} + 2\alpha + 2\bar{\beta} - \bar{\tau})T_{nn}]$$

GRAVITATIONAL WAVES

- Teukolsky equation
- 1) far region: source free, outgoing at infinity
- 2) NHEK or near-NHEK region: source stress tensor, ingoing at horizon
- Stress tensor with quadrupole correction

$$T^{\mu\nu} = \int d\tau[(p^{(\mu}u^{\nu)})\mathcal{D} + \frac{1}{3}R_{\alpha\beta\gamma}{}^{(\mu}J^{\nu)\gamma\beta\alpha}\mathcal{D} - \nabla_\alpha(S^{\alpha(\mu}u^{\nu)}\mathcal{D}) - \frac{2}{3}\nabla_\alpha\nabla_\beta(J^{\alpha(\mu\nu)\beta}\mathcal{D})]$$

- For 2^N-pole,

$$T^{\mu\nu} = \sum_{i,j,k\geq 0}^{i+j+k\leq N} T^{\mu\nu}_{ijk}\delta^{(i)}(r-r_0)\delta^{(j)}(\theta-\frac{\pi}{2})\delta^{(k)}(\phi+\alpha r_0 t)$$

- Matching at intermediate region

$$\delta\psi_4|_{Kerr} = M^2\lambda^{4/3}\times\delta\psi_4|_{NHEK}$$

GRAVITATIONAL WAVES

- Circular(l_*)

$$h_+ - ih_\times = \frac{\mu}{\hat{r}} \sum_{l,m} \mathcal{A}_{lm}\left(\frac{\ell}{\ell_*}, \chi q; \lambda, \kappa_{S^2}\right) S_{lm}(\theta) e^{im\hat{\phi} - i\hat{\omega}\hat{u}}$$

$$\mathcal{A}_{lm} = -8\frac{M^4}{qm^2} B_{lm}(x_*) \mathcal{K}_\kappa^{far}$$

$$B_{lm}(x_*) = \frac{q}{\mathcal{W}_\kappa M^4 r_0} \left\{ \frac{\mathcal{R}^{\text{in}}_{lm\omega}(r_0)}{(1+2\kappa_0)^2} \left[{}_{-2}t_4 \left(\frac{V^2(r_0)}{(1+2\kappa_0)^2} + \frac{r_0^2 V''(r_0)}{1+2\kappa_0} \right) - b_3 \frac{r_0 V'(r_0)}{1+2\kappa_0} + b_0 + b_2 \frac{V(r_0)}{1+2\kappa_0} \right] + \frac{r_0 \mathcal{R}^{\text{in}\prime}_{lm\omega}(r_0)}{(1+2\kappa_0)^2} \left[-b_1 + 2\,{}_{-2}t_4 \frac{r_0 V'(r_0)}{1+2\kappa_0} - \tilde{b}_3 \frac{V(r_0)}{1+2\kappa_0} \right] \right\}$$

$$\mathcal{K}_\kappa^{far} \equiv \frac{\lambda^h \kappa^{-h} k_1}{1 - \lambda^{2h-1} k_2 \frac{\Gamma(h-in+im)}{\Gamma(1-h-in+im)}}$$

$$n \equiv \frac{\omega}{\kappa} + m.$$

$$k_1 \equiv \frac{2^{im} e^{-im/2} \Gamma(2-2h)}{\Gamma(1-h+im-s)} (im)^{h-1+im-s} \left[1 - \frac{(-im)^{2h-1}}{(im)^{2h-1}} \frac{\sin\pi(h+im)}{\sin\pi(h-im)} \right]$$

$$k_2 \equiv (-2im)^{2h-1} \frac{\Gamma(1-2h)^2}{\Gamma(2h-1)^2} \frac{\Gamma(h-im+s)}{\Gamma(1-h-im+s)} \frac{\Gamma(h-im-s)}{\Gamma(1-h-im-s)},$$

$$h \equiv \frac{1}{2}(1 + \text{sign}(\eta^2)\eta) \qquad \eta \equiv \sqrt{1 - 7m^2 + 4\mathcal{E}}$$

GRAVITATIONAL WAVES

- Circular(l_*)

$$\mathcal{R}^{\rm in}_{lm\omega}(r) = r^{-in/2-s}(\frac{r}{2\kappa}+1)^{i(\frac{n}{2}-m)-s}{}_2F_1(h-im-s,1-h-im-s,1-in-s,-\frac{r}{2\kappa})$$

$$\mathcal{W}_\kappa \equiv -\frac{(2\kappa)^{1-h-in/2}\Gamma(2h)\Gamma(1-in-s)}{\Gamma(h+im-in)\Gamma(h-im-s)}$$

$$\begin{aligned}
b_0 &= {}_{-2}t_0 + 4\frac{1+\kappa_0}{1+2\kappa_0}{}_{-2}t_1 + 4\frac{5+10\kappa_0+6\kappa_0^2}{(1+2\kappa_0)^2}{}_{-2}t_2 + 24\frac{(1+\kappa_0)(5+10\kappa_0+8\kappa_0^2)}{(1+2\kappa_0)^3}{}_{-2}t_3 \\
&\quad +24\frac{35+140\kappa_0+252\kappa_0^2+224\kappa_0^3+80\kappa_0^4}{(1+2\kappa_0)^4}{}_{-2}t_4, \\
b_1 &= {}_{-2}t_1 + 6\frac{1+\kappa_0}{1+2\kappa_0}{}_{-2}t_2 + 2\frac{19+38\kappa_0+24\kappa_0^2}{(1+2\kappa_0)^2}{}_{-2}t_3 + 16\frac{(1+\kappa_0)(17+34\kappa_0+30\kappa_0^2)}{(1+2\kappa_0)^3}{}_{-2}t_4, \\
b_2 &= {}_{-2}t_2 + 12\frac{1+\kappa_0}{1+2\kappa_0}{}_{-2}t_3 + 2\frac{61+122\kappa_0+72\kappa_0^2}{(1+2\kappa_0)^2}{}_{-2}t_4, \\
b_3 &= {}_{-2}t_3 + 18\frac{1+\kappa_0}{1+2\kappa_0}{}_{-2}t_4, \\
\tilde{b}_3 &= {}_{-2}t_3 + 16\frac{1+\kappa_0}{1+2\kappa_0}{}_{-2}t_4.
\end{aligned}$$

GRAVITATIONAL WAVES

- Radial source term of Teukolsky equation

$$
\begin{aligned}
{}_{-2}T_{lm\tilde{\Omega}}(r) &= -4M^2 \int_0^{2\pi} d\phi e^{-im(\phi-\tilde{\omega}t)} \int_0^{\pi} d\theta \sin\theta S_{lm}(\theta)(1+\cos^2\theta)(1-i\cos\theta)^4 \mathcal{T}_4 \\
&= q \sum_{i=0}^{N+2} {}_{-2}t_i \frac{r_0^{i+3}}{M^4} \delta^{(i)}(r-r_0)
\end{aligned}
$$

- ${}_{-2}t_i$ are fixed by circular orbit

GRAVITATIONAL WAVES

- $\mathcal{A}_{lm}$ is independent of M
- $h_+ - ih_\times \propto \frac{\mu}{\hat{r}}$ typical fall off behavior
- For extreme Kerr black holes, the frequency of the emitted GWs is locked by kinematics to be extremal value $\widehat{\omega}_{ext} = \frac{m}{2M}$
- For near-extreme Kerr black holes, the frequency is relatively shifted

$$\frac{\hat{\omega} - \hat{\omega}_{ext}}{\hat{\omega}_{ext}} = -\frac{\sqrt{3}}{2}\frac{\lambda}{\sqrt{1-\frac{\ell_*^2}{\ell^2}}}\left(1+\frac{1}{2}(\kappa_{S^2}-1)\chi^2 q^2\right) + O(q^3)$$

- Near-NHEK approximation requires $\frac{\lambda}{\sqrt{1-\frac{\ell_*^2}{\ell^2}}} \ll 1$
- l *can be very close to* l_* *but can never be reached in near* $-$ *NHEK*.
- Maximal: $l \to l_*$, minimal: $l \to \infty$
- Vanishes at first order of $\chi q = \frac{S}{\mu M}$
- Vanishes at second order of S for black holes ($\kappa_{S^2} = 1$) , non-zero for neutron stars

GRAVITATIONAL WAVES

- Amplitude is independent of r_0
- The leading contribution is from the modes with $h = \frac{1}{2} - i\delta_{lm}$
- Scaling behavior in the limit $l \to l_*$

$$\lim_{\ell\to\ell_*} \mathcal{A}_{lm}\left(\frac{\ell}{\ell_*}, \chi q, \lambda, \kappa_{S^2}\right) \sim \left(\frac{\lambda}{\sqrt{1-\frac{\ell_*^2}{\ell^2}}}\right)^{1/2}$$

- Generalization of the scaling behavior with spin and higher multipole corrections.

  G.Compere, K.Fransen, T.Hertog, J.Long (2017)

- No divergent in the limit $l \to l_*$
- The orbit is completely fixed given energy and orbital angular momentum, using Boyer-Linquist coordinates $\hat{x} = \frac{\hat{r}-\hat{r}_+}{\hat{r}_+}$, $\hat{x}_0 = \frac{\lambda}{\kappa_0}$

$$\lim_{\ell\to\ell_*} \mathcal{A}_{lm}\left(\frac{\ell}{\ell_*}, \chi q, \lambda, \kappa_{S^2}\right) \sim \hat{x}_0^{1/2}$$

GRAVITATIONAL WAVES

- Energy flux (working in progress)
- Since we already obtained the waveform at infinity and horizon, the energy flux can be found to be (NHEK)
- $\dot{E}_\infty = q^2 \hat{x}_0 [a_\infty^{(0)} + a_\infty^{(1)} \chi q + \left(a_\infty^{(2)} + \kappa_{S^2} \tilde{a}_\infty^{(2)} \right) (\chi q)^2 + \cdots]$
- $\dot{E}_H = q^2 \hat{x}_0 [a_H^{(0)} + a_H^{(1)} \chi q + \left(a_H^{(2)} + \kappa_{S^2} \tilde{a}_H^{(2)} \right) (\chi q)^2 + \cdots]$
- $a_\infty^{(i)}, a_H^{(i)}$ are constants which should be evaluated numerically.
- $a_\infty^{(0)} = 0.987, a_H^{(0)} = -0.133$ S.Gralla, S.Hughes & N.Warburton (2016)
- $a_\infty^{(1)} = ?, a_H^{(1)} = ?$ First order correction from spin effect
- $a_\infty^{(2)} = ?, a_H^{(2)} = ?$ Second order correction from spin effect
- $\tilde{a}_\infty^{(2)} = ?, \tilde{a}_H^{(2)} = ?$ First order correction from size (quadrupole) effect

DISCUSSION & CONCLUSION

- Detectability
- Extremely small λ, rapidly spinning Kerr black hole
- Existence?
- K.S.Thorne bound (1974): $J \lesssim 0.998M^2$
- X-ray observing campaigns for AGNs
- L.Brenneman, "Measuring Supermassive Black Hole Spins in Active Galactic Nuclei" 2013
- Maybe we can assume the existence of high spin Kerr black hole

AGN	a	log M	$L_{\rm bol}/L_{\rm Edd}$	Host
MCG–6-30-15[a]	$\geq +0.98$	$6.65^{+0.17}_{-0.17}$	$0.40^{+0.13}_{-0.13}$	E/S0
Fairall 9[b]	$+0.52^{+0.19}_{-0.15}$	$8.41^{+0.11}_{-0.11}$	$0.05^{+0.01}_{-0.01}$	Sc
SWIFT J2127.4+5654[c]	$+0.6^{+0.2}_{-0.2}$	$7.18^{+0.07}_{-0.07}$	$0.18^{+0.03}_{-0.03}$	—
1 H0707–495[d]	$\geq +0.98$	$6.70^{+0.40}_{-0.40}$	$\sim 1.0_{-0.6}$	—
Mrk 79[e]	$+0.7^{+0.1}_{-0.1}$	$7.72^{+0.14}_{-0.14}$	$0.05^{+0.01}_{-0.01}$	SBb
Mrk 335[f]	$+0.70^{+0.12}_{-0.01}$	$7.15^{+0.13}_{-0.13}$	$0.25^{+0.07}_{-0.07}$	S0a
NGC 3783[g]	$\geq +0.98$	$7.47^{+0.08}_{-0.08}$	$0.06^{+0.01}_{-0.01}$	SB(r)ab
Ark 120[h]	$+0.94^{+0.1}_{-0.1}$	$8.18^{+0.05}_{-0.05}$	$0.04^{+0.01}_{-0.01}$	Sb/pec
3C 120[i]	≥ 0.95	$7.74^{+0.20}_{-0.22}$	$0.31^{+0.20}_{-0.19}$	S0
1 H0419–577[j]	$\geq +0.88$	$8.18^{+0.12}_{-0.12}$	$1.27^{+0.42}_{-0.42}$	—
Ark 564[j]	$+0.96^{+0.01}_{-0.06}$	≤ 6.90	≥ 0.11	SB
Mrk 110[j]	$\geq +0.99$	$7.40^{+0.09}_{-0.09}$	$0.16^{+0.04}_{-0.04}$	—
SWIFT J0501.9-3239[j]	$\geq +0.96$	—	—	SB0/a(s) pec
Ton S180[j]	$+0.91^{+0.02}_{-0.09}$	$7.30^{+0.60}_{-0.40}$	$2.15^{+3.21}_{-1.61}$	—
RBS 1124[j]	$\geq +0.98$	8.26	0.15	—
Mrk 359[j]	$+0.66^{+0.30}_{-0.54}$	6.04	0.25	pec
Mrk 841[j]	$\geq +0.52$	7.90	0.44	E
IRAS 13224-3809[j]	$\geq +0.995$	7.00	0.71	—
Mrk 1018[j]	$+0.58^{+0.36}_{-0.74}$	8.15	0.01	S0
IRAS 00521-7054[l]	$\geq +0.84$	—	—	—
NGC 4051[m]	$\geq +0.99$	6.28	0.03	SAB(rs)bc
NGC 1365[k]	$+0.97^{+0.01}_{-0.04}$	$6.60^{+1.40}_{-0.30}$	$0.06^{+0.06}_{-0.04}$	SB(s)b

DISCUSSION & CONCLUSION

- Detectability
- Leading order frequency is locked by extreme Kerr black hole
- $f_H = \frac{1}{4\pi M} = 1.6 \times 10^{-2} (\frac{10^6 M_{solar}}{M})$
- SMBH: space-based detectors, LISA
- IMBH: ground-based detectors, Advanced LIGO, VIRGO
- Precise observation?

$$\frac{\hat{\omega} - \hat{\omega}_{ext}}{\hat{\omega}_{ext}} = -\frac{\sqrt{3}}{2} \frac{\lambda}{\sqrt{1 - \frac{\ell_*^2}{\ell^2}}} (1 + \frac{1}{2}(\kappa_{S^2} - 1)\chi^2 q^2)$$

- Black holes and Neutron stars are different at second order of spin

DISCUSSION & CONCLUSION

- Detectability
- Extreme mass ratio coalescence, $q \sim 10^{-6}$, spin and size effect are too small
- Intermediate mass ratio coalescence (IMRAC), $q \sim 10^{-2}$, maybe it is more closely related to experiments.
- Two types of IMRACs

  1) stellar mass BH falls into IMBH, LIGO

  2) IMBH falls into SMBH, LISA
- The method is reliable for IMRACs?
- Existence of IMBH?
- The self force effect should be comparable to spin effect
- Convergence problem for higher multipole?

DISCUSSION & CONCLUSION

- Future direction
- 1) Self force correction
- 2) Circular orbits out of equatorial plane (spin effect is necessary)
- 3) Plunging orbits from conformal transformation
- 4) MPD equation at the full level by including all higher multipoles for BHs?
- 5) Exact critical orbital angular momentum with all higher multipole corrections?
- 6) Numerical simulation and confirm our results
- 7) …

THANKS FOR YOUR ATTENTION!

TECHNICAL DETAILS

CONFORMAL TRANSFORMATION

NHEK→ NHEK isomorphism

$$\bar{R} = \frac{R^2(1+T^2) - 1 + (1 + R^2(1-T^2))\cos\zeta - 2R^2 T \sin\zeta}{2R}$$

$$\bar{T} = \frac{2R^2 T \cos\zeta + (1 + R^2(1-T^2))\sin\zeta}{2R}\frac{1}{\bar{R}} \qquad (10)$$

$$\bar{\Phi} = \Phi + \log\frac{\cos\frac{\zeta}{2} R - \sin\frac{\zeta}{2}(1 + RT)}{\cos\frac{\zeta}{2} R - \sin\frac{\zeta}{2}(-1 + RT)}$$

NHEK→ NHEK discrete transformation

$$T \to -T, \ \Phi \to -\Phi \qquad (11)$$

TECHNICAL DETAILS

CONFORMAL TRANSFORMATION

near-NHEK$\rightarrow$ near-NHEK isomorphism

$$
\begin{aligned}
r &= (\bar{r}+\kappa)\cos\zeta - \sqrt{\bar{r}(\bar{r}+2\kappa)}\sin\zeta\sinh\kappa\bar{t} - \kappa, \\
t &= \frac{1}{2\kappa}\log\frac{\sqrt{\bar{r}(\bar{r}+2\kappa)}(\cosh\kappa\bar{t}+\cos\zeta\sinh\kappa\bar{t})+\sin\zeta(\bar{r}+\kappa)}{\sqrt{\bar{r}(\bar{r}+2\kappa)}(\cosh\kappa\bar{t}-\cos\zeta\sinh\kappa\bar{t})-\sin\zeta(\bar{r}+\kappa)}, \\
\phi &= \bar{\phi}+\frac{1}{2}\log\frac{r}{r+2\kappa}+\log\frac{e^{\kappa\bar{t}}\sqrt{\bar{r}+2\kappa}\cos\frac{\zeta}{2}+\sqrt{\bar{r}}\sin\frac{\zeta}{2}}{e^{\kappa\bar{t}}\sqrt{\bar{r}}\cos\frac{\zeta}{2}+\sqrt{\bar{r}+2\kappa}\sin\frac{\zeta}{2}}.
\end{aligned}
$$

near-NHEK$\rightarrow$ near-NHEK discrete transformation

$$t \rightarrow -t, \phi \rightarrow -\phi \qquad (12)$$

TECHNICAL DETAILS
CONFORMAL TRANSFORMATION

NHEK$\to$ near-NHEK diffeomorphism

$$\begin{aligned} r &= \kappa(-RT-1), \\ t &= \frac{1}{\kappa}\log\frac{R}{\sqrt{R^2T^2-1}}, \\ \phi &= \Phi + \frac{1}{2}\log\frac{-RT-1}{-RT+1}. \end{aligned} \tag{13}$$

near-NHEK$\to$ NHEK diffeomorphism

$$\begin{aligned} R &= \frac{1}{\kappa}e^{\kappa t}\sqrt{r(r+2\kappa)}, \\ T &= -e^{-\kappa t}\frac{r+\kappa}{\sqrt{r(r+2\kappa)}}, \\ \Phi &= \phi - \frac{1}{2}\log\frac{r}{r+2\kappa}. \end{aligned} \tag{14}$$

TECHNICAL DETAILS
NHEK, NEAR-NHEK & KERR

Energy and Angular momentum

- $\hat{e}, \hat{l}$ asymptotically flat observer
- E, L NHEK
- e, l near-NHEK

$\hat{l} = L,\ \hat{e} = \frac{\lambda^{\frac{2}{3}}}{2M} E - \frac{1}{2M} L$

$\hat{l} = l,\ \hat{e} = \frac{\lambda}{2M\kappa} e - \frac{1}{2M} l$

$\hat{x} \to 0$ region should be attached to $R \to \infty$ or $r \to \infty$ region

wave function should be matched

$$\psi(\hat{x} \to 0)|_{Kerr} \sim \psi(R \to \infty)|_{NHEK} \quad (8)$$

$$\psi(\hat{x} \to 0)|_{Kerr} \sim \psi(r \to \infty)|_{near-NHEK} \quad (9)$$

TECHNICAL DETAILS
TEUKOLSKY EQUATION

- Angular part

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\frac{\mathrm{d}S_{lm}}{\mathrm{d}\theta})+[\frac{m^2}{4}\cos^2\theta-ms\cos\theta-(\frac{m^2+2ms\cos\theta+s^2}{\sin^2\theta})+\mathcal{E}_{lm}]S_{lm}=0.$$

- Radial part

$$(r(r+2\kappa))^{-s}\frac{\mathrm{d}}{\mathrm{d}r}((r(r+2\kappa))^{s+1}\frac{\mathrm{d}R_{lm\omega}}{\mathrm{d}r})-V(r)R_{lm\omega}(r)=T_{lm\omega}(r)$$

$$V(r)=-\frac{3}{4}m^2-s(s+1)+\mathcal{E}_{lm}-2ism+\frac{(mr+\kappa n)(\kappa(2si-n)+r(2si-m))}{r(r+2\kappa)}$$

TECHNICAL DETAILS
TEUKOLSKY EQUATION

- Radial equation with Delta function

$$A(r)(B(r)R(r)')' - V(r)R(r) = T(r)$$

$$T(r) = \sum_{i=0}^{N+2} a_i \delta^{(i)}(r-r_0)$$

- Assume $R_{1,2}(r)$ are two independent solutions of homogeneous equation

$$R(r) = X_1 R_1(r)\Theta(r_0-r) + X_2 R_2(r)\Theta(r-r_0) + Y_1 R_1(r) + Y_2 R_2(r) + \sum_{i=0}^{n} \beta_i \delta^{(i)}(r-r_0)$$

- Define Wronskian

$$W = B(R_1 R_2' - R_2 R_1')$$

$$
\begin{aligned}
X_1 &= \frac{-1}{A^5B^3W}[a_0(-A^4B^3R_2) + a_1A^3B^3(AR_2' - A'R_2) + a_2A^2B^2(R_2(-AV - 2BA'^2 \\
&\quad +ABA'') + R_2'(2ABA' + A^2B')) + a_3AB(R_2(-4ABVA' - 6A'^3B^2 - 2A^2B'V \\
&\quad +A^2BV' + 6AA'A''B^2 - A^2B^2A^{(3)}) + R_2'(A^2BV + 6AB^2A'^2 + 3A^2BA'B' + 2A^3B'^2 \\
&\quad -3A^2B^2A'' - A^3BB'')) + a_4(R_2(-A^2BV^2 - 18AB^2A'^2V - 24A'^4B^3 - 11A^2BA'B'V \\
&\quad -6A^3B'^2V + 6A^2B^2A'V' + 3A^3BB'V' + 7A^2B^2A''V + 36AB^3A'^2A'' - 6A^2B^3A''^2 \\
&\quad +3A^3BB''V - A^3B^2V'' - 8A^2B^3A'A^{(3)} + A^3B^3A^{(4)}) + R_2'(6A^2B^2A'V + 24AB^3A'^3 \\
&\quad +4A^3BB'V + 12A^2B^2A'^2B' + 8A^3BA'B'^2 + 6A^4B'^3 - 2A^3B^2V' - 24A^2B^3A'A'' \\
&\quad -6A^3B^2B'A'' - 4A^3B^2A'B'' - 6A^4BB'B'' + 4A^3B^3A^{(3)} + A^4B^2B^{(3)}))], \qquad \text{(B.4)} \\
X_2 &= X_1(R_2 \leftrightarrow R_1,\ \text{keeping } W \text{ unflipped}), \qquad \text{(B.5)} \\
\beta_0 &= \frac{1}{A^3B^3}(a_2A^2B^2 + a_3AB(2AB' + 3A'B) + a_4(ABV + AB(8A'B' - 6A''B) \\
&\quad +6A^2B'^2 - 3A^2BB'' + 12A'^2B^2))), \qquad \text{(B.6)} \\
\beta_1 &= \frac{1}{A^2B^2}(a_3AB + (3AB' + 4A'B)a_4), \qquad \text{(B.7)} \\
\beta_2 &= \frac{a_4}{AB}. \qquad \text{(B.8)}
\end{aligned}
$$