Covariant Phase Space with Boundaries

Jie-qiang Wu (MIT)

Based on work in progress
with Daniel Harlow

Tsinghua Sanya International Mathematics Forum

Jan 8th, 2019

- In general relativity, covariant phase space method was developed by Iyer, Lee, Wald, and Zoupas to study Hamiltonian and black hole first law without breaking covariance
- In this work, we study the covariant phase space with more careful treatment of the boundary terms
- With this formalism, we give an explicit algorithm to calculate the Hamiltonian (without dealing with the B term $\delta \int_\partial \xi \cdot B = \int_\partial \xi \cdot \Theta(\phi, \delta\phi)$)
- To understand the covariant phase space method, we study the phase space and the symplectic form for JT gravity
- With this symplectic form, we give an explanation for the traversable wormhole
- In this work, we only focus on **classical mechanics**

Outline

Classical mechanics

- In classical mechanics, the Hamiltonian formalism is defined by phase space, Hamiltonian, and the Possion bracket or Dirac bracket (symplectic form)
- The phase space and symplectic form include everything in classical mechanics
- In statistical mechanics, the microscopic state is the volume of the phase space
- In quantum mechanics, the classical phase space is the first step of canonical quantization

Hamiltonian vs general relativity

- Hamiltonian formalism is not convenient to describe general relativity
- Hamiltonian formalism: a special coordinate and time direction
- General relativity: diffeomorphism symmetry
- Covariant phase space method

  Lee, Wald J.Math.Phys31 725(1990)

  Iyer, Wald gr-qc/9403028 gr-qc/9503052 Wald Zoupas gr-qc/9911095

  Phase space: (up to gauge equivalence,) every solution for the equation of motion corresponds to one point in the phase space

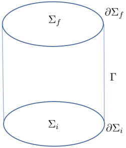
  Symplectic form: derived from the action

Simple example: point particle

- Point particle $S = \int_{t_i}^{t_f} dt \frac{1}{2}\dot{x}^2$
- Taking a variation
  $\delta S = \int_{t_i}^{t_f} dt(-\ddot{x})\delta x(t) + \dot{x}\delta x\mid_f - \dot{x}\delta x\mid_i$
- The pre-symplectic potential is defined as the initial (or final) surface term under variation of the action; 1-form in configuration space
  The pre-symplectic form is the derivative of symplectic potential in configuration space; 2-form in configuration space
- Symplectic potential: $\theta[x,\delta x] = \dot{x}(t)\delta x(t) = p\delta x$
  Symplectic form
  $\omega[x,\delta_1 x,\delta_2 x] = \delta_1\theta[x,\delta_2 x] - \delta_2\theta[x,\delta_1 x] = \delta_1 p\delta_2 x - \delta_2 p\delta_1 x$
  Hamiltonian $H = \theta[x,\dot{x}] - L = \frac{1}{2}p^2$
- Hamiltonian equation $\delta H = \omega[x,\delta x,\delta_2 x = \dot{x}]$
  or $(\delta H)_A = \omega_{AB}\xi^B$, where ξ is a flow in phase space

GR

- For more complicated theories in higher dimension even with gauge symmetry, the prescription still works
- Action: $S = \int L^{(n)} + \int_\gamma l^{(n-1)}$
  Diffeomorphism symmetry:
  $\delta\phi = \mathcal{L}_\xi \phi$, $\delta L^{(n)} = \mathcal{L}_\xi L^{(n)}$ $\delta l^{(n-1)} = \mathcal{L}_\xi l^{(n-1)}$
  ξ is parallel to the boundary
  ϕ denote all of the fields including the matter fields and metric



- Taking a variation for the action

$$\delta L^{(n)} = E^{(n)}(\phi)\delta\phi + d\Theta^{(n-1)}(\phi,\delta\phi)$$
$$-\Theta^{(n-1)}(\phi,\delta\phi) + \delta I^{(n-1)} = F^{(n-1)}(\phi)\delta\phi + dC^{(n-2)}(\phi,\delta\phi)$$

$$\begin{aligned}\delta S &= \int E^{(n)}(\phi)\delta\phi + \int_\partial F^{(n-1)}(\phi)\delta\phi \\ &+\left(\int_{\Sigma^{(n-1)},f} \Theta^{(n-1)}(\phi,\delta\phi) + \int_{\partial\Sigma^{(n-2)},f} C^{(n-2)}(\phi,\delta\phi)\right) \\ &-\left(\int_{\Sigma^{(n-1)},i} \Theta^{(n-1)}(\phi,\delta\phi) + \int_{\partial\Sigma^{(n-2)},i} C^{(n-2)}(\phi,\delta\phi)\right)\end{aligned}$$

- Pre-symplectic potential
  $\Theta_{\text{tot}}[\phi,\delta\phi] = \int_{\Sigma^{(n-1)}} \Theta^{(n-1)}(\phi,\delta\phi) + \int_{\partial\Sigma^{(n-2)}} \mathbf{C^{(n-2)}}(\phi,\delta\phi)$
  Pre-symplectic form
  $$\begin{aligned}\Omega_{\text{tot}} &= \delta_1\Theta_{\text{tot}}(\phi,\delta_2\phi) - \delta_2\Theta_{\text{tot}}(\phi,\delta_1\phi) \\ &= \int_{\Sigma^{(n-1)}} \omega^{(n-1)}(\phi,\delta_1\phi,\delta_2\phi) \\ &+ \int_{\partial\Sigma^{(n-2)}} (\delta_\mathbf{1}\mathbf{C^{(n-2)}}(\phi,\delta_\mathbf{2}\phi) - \delta_\mathbf{2}\mathbf{C^{(n-2)}}(\phi,\delta_\mathbf{1}\phi))\end{aligned}$$
- Compared with Wald, we have an extra boundary term related to $C^{(n-2)}$

  Iyer, Wald gr-qc/9403028 gr-qc/9503052
- In Einstein-Hilbert action $C \sim \delta g_{ab} n^a \tau^b$
- In Einstein-Hilbert action, JT gravity, $f(R)$ gravity, Lovelock gravity, the C term vanish if we choose the gauge that the foliation is orthogonal to the boundary
- Non-zero C term: $S = \int \nabla_a R_{bc} \nabla^a R^{bc}$
- It is convenient to keep the gauge redundancy and non-zero C term in calculation

Hamiltonian

- Noether current: $j_\xi^{(n-1)} = \Theta^{(n-1)}[\phi, \mathcal{L}_\xi \phi] - \xi \cdot L^{(n)}$
- Noether charge: $dj_\xi^{(n-1)} = 0 \quad j_\xi^{(n-1)} = dQ_\xi^{(n-2)}$
  (under on-shell condition)
- Relation with symplectic form current:
  $\delta j_\xi^{(n-1)} = \omega^{(n-1)}(\phi, \delta\phi, \mathcal{L}_\xi \phi) + d(\xi \cdot \Theta^{(n-1)})$
  $\int_\Sigma \omega(\phi, \delta\phi, \mathcal{L}_\xi \phi) = \int_{\partial\Sigma}(\delta Q_\xi - \xi \cdot \Theta(\phi, \delta\phi))$
- Boundary action variation
  $-\Theta^{(n-1)}(\phi, \delta\phi) + \delta I^{(n-1)} = F^{(n-1)}(\phi)\delta\phi + dC^{(n-2)}(\phi, \delta\phi)$
- Hamiltonian
  $\int_\Sigma \omega^{(n-1)}(\phi, \delta\phi, \mathcal{L}_\xi \phi) + \int_{\partial\Sigma} \delta C^{(n-2)}(\phi, \mathcal{L}_\xi \phi) - \mathcal{L}_\xi C^{(n-2)}(\phi, \delta\phi)$
  $= \int_{\partial\Sigma} \delta(Q_\xi^{(n-2)} + C^{(n-2)}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot I^{(n-1)})$
- Hamiltonian equation $\Omega_{\text{tot}}[\phi, \delta\phi, \mathcal{L}_\xi \phi] = \delta H_\xi$
  $H_\xi = \int_{\partial\Sigma}(Q_\xi + C^{(n-2)}(\phi, \mathcal{L}_\xi \phi) - \xi \cdot I^{(n-1)})$

- Hamiltonian ($Q^{(n-2)} = dj^{(n-1)}$)
$$H_\xi = \int_{\partial\Sigma}(Q_\xi + C^{(n-2)}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot l^{(n-1)})$$
$$= \int_\Sigma \Theta^{(n-1)}(\phi, \delta\phi, \mathcal{L}_\xi\phi) + \int_{\partial\Sigma} C^{(n-2)}(\phi, \delta\phi, \mathcal{L}_\xi\phi)$$
$$- \int_\Sigma \xi \cdot L^{(n)}(\phi) - \int_{\partial\Sigma} \xi \cdot l^{(n-1)}(\phi)$$
- Classical mechanics: $H = p\dot{q} - L$

Hawking, Horowitz gr-qc/9501014

- Ambiguity I:
$$S = \int L^{(n)} + \int_\Gamma l^{(n-1)} \Rightarrow L \to L + dX \quad l \to l + X$$
Ambiguity II:
$$\delta L^{(n)} = E^{(n)}(\phi)\delta\phi + d\Theta^{(n-1)}(\phi, \delta\phi)$$
$$-\Theta^{(n-1)}(\phi, \delta\phi) + \delta l^{(n-1)} = F^{(n-1)}(\phi)\delta\phi + dC^{(n-2)}(\phi, \delta\phi)$$
$$\Rightarrow \Theta \to \Theta + dY \quad C \to C - Y$$
The Hamiltonian have no ambiguities

Relation with Brown York tensor

- In Einstein-Hilbert action, JT gravity, H_ξ in our algorithm matches with the Brown York tensor's calculation
- A general proof:
- Taking a variation $\delta\phi = \mathcal{L}_\xi \phi \quad \xi \mid_\partial \neq 0$
$$\delta S = \Theta_{\mathrm{tot},f}[\phi, \mathcal{L}_\xi\phi] - \Theta_{\mathrm{tot},i}[\phi, \mathcal{L}_\xi\phi] + \int_\gamma \sqrt{-\gamma}\nabla_a \xi_b T^{ab}$$
$$= (\Theta_{\mathrm{tot},f}[\phi, \mathcal{L}_\xi\phi] - \int_{\partial\Sigma_f} dx^{n-2}\sqrt{h}\lambda_a\xi_b T^{ab})$$
$$-(\Theta_{\mathrm{tot},i}[\phi, \mathcal{L}_\xi\phi] - \int_{\partial\Sigma_i} dx^{n-2}\sqrt{h}\lambda_a\xi_b T^{ab})$$
- Diffeomorphism symmetry $\delta L = \mathcal{L}_\xi L \quad \delta l = \mathcal{L}_\xi l$
$$\delta S = \int \mathcal{L}_\xi L^{(n)} + \int_\gamma \mathcal{L}_\xi l^{(n-1)}$$
$$= (\int_{\Sigma_f} \xi \cdot L^{(n)} + \int_{\partial\Sigma_f} \xi \cdot l^{(n-1)}) - (\int_{\Sigma_i} \xi \cdot L^{(n)} + \int_{\partial\Sigma_i} \xi \cdot l^{(n-1)})$$
- Compare the two equations
$$H_\xi = \Theta_{\mathrm{tot}}[\phi, \mathcal{L}_\xi\phi] - (\int_\Sigma \xi \cdot L + \int_{\partial\Sigma} \xi \cdot l)$$
$$= \int_{\partial\Sigma_f} dx^{n-2}\sqrt{h}\lambda_a\xi_b T^{ab}$$

Black hole first law

- The C term don't change black hole first law
  $$\begin{aligned}\delta H_\xi &= \int_\Sigma \omega(\phi,\delta\phi,\mathcal{L}_\xi\phi) + \int_{\partial\Sigma}(\delta C(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi C(\phi,\delta\phi)) \\ &= \int_{\partial\Sigma}(\delta Q_\xi - \xi\cdot\Theta(\phi,\delta\phi)) + \int_{\partial\Sigma}(\delta C(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi C(\phi,\delta\phi))\end{aligned}$$
- Under stationary black hole background $\mathcal{L}_\xi\phi = 0$, the C related term vanish
  $$\int_{\partial\Sigma}(\delta C(\phi,\mathcal{L}_\xi\phi) - \mathcal{L}_\xi C(\phi,\delta\phi)) = 0$$
  The first law goes back to Wald's derivation
  Iyer, Wald gr-qc/9403028

Gauge invariance

- When $\xi\mid_{\partial}=0$, $H_\xi = 0$
- H_ξ only depend on $\xi\mid_{\partial}$ so is gauge invariant
  $H_\xi = \int_{\partial\Sigma}(Q_\xi + C^{(n-2)}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot l^{(n-1)})$
- Criteria of gauge invariance of Θ_{tot} and Ω_{tot}:
  $\Theta_{\text{tot}}[\phi, \mathcal{L}_\xi\phi] = 0 \quad \Omega_{\text{tot}}[\phi, \delta\phi, \mathcal{L}_\xi\phi] = 0 \quad (\xi\mid_{\partial}=0)$
- $\Omega_{\text{tot}}[\phi, \delta\phi, \mathcal{L}_\xi\phi] = \delta H_\xi = 0$
  Ω_{tot} is gauge invariant
- $\Theta_{\text{tot}}[\phi, \mathcal{L}_\xi\phi] - \int \xi \cdot L - \int \xi \cdot l = H_\xi = 0$
  Θ_{tot} is gauge invariant if and only if the bulk Lagrangian density vanish under on-shell condition

Symplectic form

- To have a better understanding for covariant phase space, we explicitly build the phase space and calculate the symplectic form in JT gravity

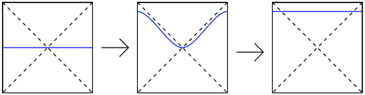
Pure JT gravity

- JT gravity $S=\int dxdt\sqrt{-g}\Phi(R+2)+\int dt\sqrt{-\gamma}\Phi(K-1)$

  Almheiri, Polchinski 1402.6334 Maldacena, Stanford, Yang 1606.01857
- The bulk Lagrangian density vanish under on-shell condition, so the symplectic potential is gauge invariant
- Solutions: $\Phi=c\frac{1-uv}{1+uv}$ $\quad ds^2=-\frac{dudv}{(1+uv)^2}$

  Boundary condition: $\Phi=\frac{\phi_0}{\epsilon}$ $\quad ds^2=-\frac{d\rho^2}{\epsilon^2}$

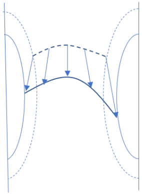
  AdM charge: $H=\frac{c^2}{\phi_0}$
- Cauchy surface



  The configuration depends on the ends of Cauchy surface
- The configuration space can be described by $(c,\rho_{L,0},\rho_{R,0})$ (Gauge redundancy)

- To study the symplectic potential, we take a variation of the solution and also the boundary of Cauchy surface

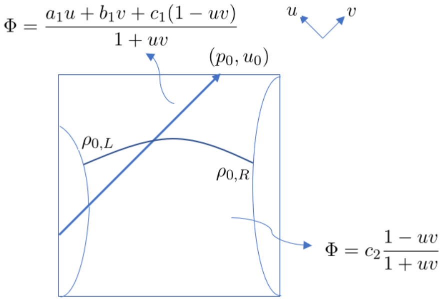
- In (u, v) coordinate $\delta^0 g_{ab} = 0 \quad \delta^0\Phi \neq 0$
- The boundary and the Cauchy surface also change
- To compare the two configurations, we need to pull the second Cauchy surface back to the first one
$\delta g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a \quad \delta\Phi = \delta^0\Phi + \mathcal{L}_\xi \Phi$



- The symplectic potential only depend on δg_{ab} not on $\delta\Phi$
  $\Theta_{\text{tot}}[g_{\mu\nu},\Phi;\delta g_{\mu\nu},\delta\Phi]$
  $= \int_\Sigma \sqrt{\sigma}(-1)t_\rho[g^{\mu\rho}\Phi\nabla^\nu\delta g_{\mu\nu} - g^{\nu\rho}\nabla^\mu\Phi\delta g_{\mu\nu}$
  $-\Phi\nabla^\rho(g^{\mu\nu}\delta g_{\mu\nu}) + \nabla^\rho\Phi g^{\mu\nu}\delta g_{\mu\nu}] + \sum_i \Phi u^\lambda n^\rho \delta g_{\rho\lambda}\,|_{\partial_i}$
- When $\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \quad \delta\Phi \neq 0$
  $\Theta_{\text{tot}} = \sum_i 2[-\Phi u_\rho K^{\rho\nu}\xi_\nu + n^\nu\nabla_\nu\Phi u_\rho\xi^\rho$
  $+\Phi u^\lambda D_\lambda(n^\nu\xi_\nu) - u^\rho\nabla_\rho\Phi n_\nu\xi^\nu]\,|_{\partial_i}$
  $\Theta_{\text{tot}} = \sum_i 2[\frac{c^2}{\phi_0}\delta\rho_{0,i} + \frac{c}{\phi_0}\rho_{0,i}\delta c]$
  $\Omega_{\text{tot}} = \delta\Theta_{\text{tot}} = 2\frac{c}{\phi_0}\delta c\wedge\delta(\rho_{0,R}+\rho_{0,L}) = \delta H\wedge\delta(\rho_{0,L}+\rho_{0,R})$
- Phase space $(H,\rho_{0,L})$, sympletic form $\Omega = \delta H\wedge\delta\rho_{0,L}$
  Harlow, Jafferis 1804.01081
- Geometric description of $\rho_{0,L}$:
  We start from the right end of Cauchy surface, and shoot in a geodesic orthogonal to the boundary. It touches the left boundary. The relative distance to the left end of Cauchy is $\rho_{0,L}$

JT gravity with one particle

- We consider JT gravity coupled with one massless particle
  $S = S_{\text{JT}} + \int_{\text{world line}} d\lambda \frac{1}{2} e(\lambda) g_{ab}(y(\lambda)) \frac{\partial y^a}{\partial\lambda}\frac{\partial y^b}{\partial\lambda}$
- Phase space $(p_0, u_0, c_2, \rho_{0,R})$



- Inverse of symplectic potential, $A, B = (p_0, u_0, c_2, \rho_{0,R})$

$$\Omega^{AB} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{2\phi_0 p_0}{c_2^2}(\log u_0 + 1) & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -8\pi G\frac{\phi_0}{c_2} \\ -\frac{2\phi_0 p_0}{c_2^2}(\log u_0 + 1) & 0 & 8\pi G\frac{\phi_0}{c_2} & 0 \end{pmatrix}$$

for small u_0

- Traversable wormhole
  Hamiltonian equation $\Omega^{AB}(\delta X)_B = \xi^A$
  $H_R = \frac{c_2^2}{\phi_0}$ $H_L = \frac{c_2^2+2p_0u_0c_2}{\phi_0}$ don't generate traversable wormhole
  $X = f(p_0)\mathcal{O}((u_0)^0)$ generate traversable wormhole
  $\langle\psi_L\psi_R\rangle \sim \frac{1}{\epsilon^2}e^{-L}$ belongs to this class
  Gao, Jafferis, Wall 1608.05687 Maldacena, Stanford, Yang 1704.05333

Conclusion

- In this work, we study the covariant phase space with more careful treatment of the boundary terms
- With this formalism, we give an algorithm to calculate the Hamiltonian
- With the covariant phase space method, we study the phase space and symplectic form for pure JT gravity and JT gravity coupled with one point particle
- As a cross check, we re-derive the traversable wormhole effect

Open question

For Hamiltonian:

- H_ξ when ξ is not parallel to the boundary
- Conserved quantity defined at null infinity
  Wald Zoupas qc/9911095
  A definition with finite IR cut-off?
- The inner boundary: horizon
- Gravity's modular Hamiltonian;
  a direct proof of JLMS formula
  Jafferis, Lewkowycz, Maldacena, Suh 1512.06431
  Dong, Harlow, Marlof 1811.05382
- A measure for C term;
  black hole first law $\rightarrow$ entanglement entropy first law

Open question

For JT gravity phase space:

- Multi-particle case; simplification in kinematics; $SL(2)$ charge
- Relation with Swarzian Yang 1809.08647
  Kitaev, Suh, 1711.08467 1808.07032
- A microscopic counting of the variation of near extremal black hole entropy δS

Thanks

Thanks for your attention!