Structure Constants from Modularity in Warped Conformal Theories

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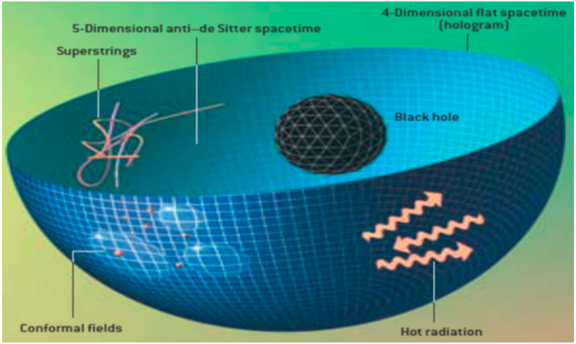
Introduction

- The holographic dualities, which relate a quantum theory of gravity to a quantum field theory without gravity in fewer dimensions, play essential roles in theoretical physics.
- The benchmark of the holographic dualities is the AdS/CFT correspondence established based on string theory [J. Maldacena, 1998].



- However, the existence of holographic dualities is not contingent on the validity of string theory.
- The Asymptotic Symmetry Group (ASG) method is successfully applied to AdS_3/CFT_2 without invoking string theory [J. D. Brown, M. Henneaux, 1986].
- The study of holography also goes beyond the standard AdS/CFT correspondence.
- The main reason behind these expectation is that the entropy of black holes is given by the area instead of volume in a general form,

$$S_{BH} \sim \frac{Area}{\ell_p^{d-1}} \,. \tag{1}$$

- It is necessary to extend the idea of holography to non-AdS case in order to understand the quantum gravity on a complete level. The efforts come from many aspects:
- The scaling limit of near horizon geometry of Kerr black holes has enlarged $SL(2,R)_L \times U(1)_R$ isometry [J. Bardeen, G. Horowitz, 1999].
- Kerr/CFT claims that the extremal Kerr black holes are described by a chiral half of a two dimensional CFT [M. Guica, T. Hartman, W. Song, A, Strominger, 2009].

$$c_L = \frac{12J}{\hbar}, \quad T_L = \frac{1}{2\pi}\,. \tag{2}$$

- The enhancement of the $U(1)_R$ isometry to the full Virasoro algebra.
- The perfect match between the black hole entropy formula and the Cardy entropy.

- The part of the geometry (at fixed polar angle) that appears to play the key role in the duality is a warped AdS_3 (WAdS) factor.

$$da^2 = 2GJ\Omega(\theta)^2 \left(\frac{-dt^2 + dy^2}{y^2} + d\theta^2 + \Lambda(\theta)^2 (d\phi + \frac{dt}{y})^2 \right) \tag{3}$$

- The structure of WAdS is that of a fibration (with warping factor multiplying the fiber metric) of a real line over AdS_2.
- The warping factor along the fiber breaks the $SL(2,R)_L \times SL(2,R)_R$ isometry group of AdS_3 down to $SL(2,R) \times U(1)$.
- In topological massive gravity (TMG), the AdS_3 vacua is unstable due to the negative energy of massive excitations ($G > 0, \mu > 0$) [S. Deser, R. Jackiw, S. Templeton, 1982].

- For generic $\mu\ell$, WAdS as possibly stable vacua and various type of warped black holes are found in TMG [D. Anninos, W. Li, M. Padi, W. Song, A. Strominger, 2009].
- For $\mu\ell > 3$, the WAdS is said to be stretched and there exist regular black holes which are asymptotic to WAdS with a spacelike $U(1)$.
- These regular black holes are shown to be discrete quotients of WAdS just as BTZ black holes are discrete quotients of ordinary AdS_3.
- Further more, given the left and right moving temperature during quotients, the warped black hole entropy matches the Cardy entropy provided

$$c_L = \frac{12\mu\ell^2}{G((\mu\ell)^2+27)}, \quad c_R = \frac{15(\mu\ell)^2+81}{G\mu((\mu\ell)^2+27)}. \tag{4}$$

So $\mu\ell > 3$ TMG is conjectured to be dual to a 2D CFT.

- The asymptotic symmetry analysis for spacelike stretched WAdS and the consistent boundary conditions are also presented [G. Compère, S. Detournay, 2008,2009].
- The asymptotic algebra they get is a Virasoro algebra and a current algebra, which indicate that dual field theory would have symmetry other than conformal symmetry.
- The Virasoro Kac-Moody algebra also shows up in the asymptotic symmetry analysis for AdS_3 with mixed chiral boundary conditions (CSS B.C.) [G. Compère, W. Song, A. Strominger, 2013].

Known results of Warped CFT

- A two dimensional quantum field theory with two global translational symmetries and a chiral global scaling symmetry have an extended local Virasoro plus $U(1)$ Kac-Moody algebra [D. Hofman, A. Strominger, 2011].

$$x^- \to x^- + a, \;\; x^+ \to x^+ + b, \;\; x^- \to \lambda x^- \tag{5}$$

- A warped conformal field theory is characterized by the warped conformal symmetry. The global symmetries are $SL(2,R) \times U(1)$, while the local symmetry algebra is a Virasoro algebra plus a $U(1)$ Kac-Moody algebra.

- In position space, a general warped conformal symmetry transformation can be written as

$$x'^{-} = f(x^{-}), \quad x'^{+} = x^{+} + g(x^{-}), \tag{6}$$

where $f(x^{-})$ and $g(x^{-})$ are two arbitrary functions.

- Consider a WCFT on a plane, denote $T(x^{-})$ and $P(x^{-})$ as the Noether currents associated with translations along x^{-} and x^{+}, the conserved charges

$$L_n = -\frac{i}{2\pi}\int \mathrm{d}x x^{n+1} T(x), \quad P_n = -\frac{1}{2\pi}\int \mathrm{d}x x^{n} P(x)\,. \tag{7}$$

form a Virasoro Kac-Moody algebra,

$$\begin{aligned}
[L_n, L_m] =& (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m}\,,\\
[L_n, P_m] =& -mP_{n+m}\,,\\
[P_n, P_m] =& k\frac{n}{2}\delta_{n,-m}\,,
\end{aligned} \tag{8}$$

- Some specific examples of WCFT:

  chiral Liouville gravity [G. Compère, W. Song, A. Strominger, 2013]

  Weyl fermion models [D. Hofman, B. Rollier, 2015]

  free scalar models [K. Jensen, 2017]

  CSYK as broken symmetry of WCFT [P. Chaturvedi, Y. Gu, W. Song, B. Yu, 2018].

- Under the warped conformal transformation, a primary field transforms as an h-form under Virasoro and a scalar under $U(1)$, [W. Song, JX, 2017]

$$\phi'(x'^-, x'^+) = \left(\frac{\partial x'^-}{\partial x^-}\right)^{-h} \phi(x^-, x^+) \,. \tag{9}$$

- The correlation functions of WCFT are given by,

$$\begin{aligned}
\langle\phi_1\phi_2\rangle &= \left(e^{i\sum_j q_j x_j^+}\delta_{\sum_k q_k}\right)\frac{\delta_{h_1,-h_2}}{(x_{12}^-)^{2h_1}} \\
\langle\phi_1\phi_2\phi_3\rangle &= \left(e^{i\sum_j q_j x_j^+}\delta_{\sum_k q_k}\right)\frac{C_{123}}{(x_{12}^-)^{h_1+h_2-h_3}(x_{23}^-)^{h_2+h_3-h_1}(x_{31}^-)^{h_3+h_1-h_2}} \\
\langle\phi_1\phi_2\phi_3\phi_4\rangle &= \left(e^{i\sum_j q_j x_j^+}\delta_{\sum_k q_k}\right)\left(\frac{x_{24}^-}{x_{14}^-}\right)^{h_{12}}\left(\frac{x_{14}^-}{x_{13}^-}\right)^{h_{34}}\frac{G(z)}{(x_{12}^-)^{h_1+h_2}(x_{34}^-)^{h_3+h_4}} \,.
\end{aligned}$$

- The Rényi entropy for an interval $\mathcal{D}$ can be written as,

$$S_n = \frac{1}{1-n}\log\frac{\mathrm{tr}(\rho_{\mathcal{D}}^n)}{(\mathrm{tr}\rho_{\mathcal{D}})^n} = \frac{1}{1-n}\log\frac{\langle\Phi_n(x_1)\Phi_n^\dagger(x_2)\rangle_{\mathcal{C}}}{\langle\Phi_1(x_1)\Phi_1^\dagger(x_2)\rangle_{\mathcal{C}}^n} \quad (10)$$

Here in the first equality, the Rényi entropy is related to the nth power of the reduced density matrix $\rho_{\mathcal{D}}$ for $\mathcal{D}$. This can be realized as a path integral an a manifold $\mathcal{R}_n$ which is made up of n decoupled copies of the original space $\mathcal{R}_1$. In the second equality, Φ_n is the twist field inserted at the endpoints of the interval that enforce the replica boundary conditions on a plane $\mathcal{C}$. $X_{1,2}$ are the endpoint coordinates of the interval $\mathcal{D}$.

- The expectation value of the current $T(x)$ and $P(x)$ on $\mathcal{R}_n$ can be calculated either using twist field ward identity or using Rindler transformation,

$$\frac{\langle T(x)\Phi_n(x_1)\Phi_n^\dagger(x_2)\rangle_{\mathcal{C}}}{\langle \Phi_n(x_1)\Phi_n^\dagger(x_2)\rangle_{\mathcal{C}}} = \langle T(x^{(i)})\rangle_{\mathcal{R}_n} = \frac{\mathrm{tr}(T(x)\rho_{\mathcal{D}}^n)}{\mathrm{tr}(\rho_{\mathcal{D}}^n)} = \frac{\mathrm{tr}(U^\dagger T(x)U\rho_{\mathcal{H}}^n)}{\mathrm{tr}(\rho_{\mathcal{H}}^n)},$$

where U stands for a unitary transformation inspired by Rindler transformation,

$$\frac{\tanh\frac{\pi x^-}{\beta}}{\tanh\frac{\Delta x^- \pi}{2\beta}} = \tanh\frac{\pi\tilde{x}^-}{\kappa}, \quad x^+ + \left(\frac{\bar{\beta}}{\beta} - \frac{\alpha}{\beta}\right)x^- = \tilde{x}^+ + \left(\frac{\bar{\kappa}}{\kappa} - \frac{\alpha}{\kappa}\right)\tilde{x}^- .$$

Compere two sides, we can get the expressions for the conformal dimension and charge of the twist field,

$$h_n = n\left(\frac{c}{24} + \frac{\mathcal{L}_0^{vac}}{n^2} - \frac{i\mathcal{P}_0^{vac}\alpha}{2n\pi} - \frac{\alpha^2 k}{16\pi^2}\right), \quad q_n = n\left(\frac{\mathcal{P}_0^{vac}}{n} - i\frac{k\alpha}{4\pi}\right).$$

The Rényi entropy,

$$S_n = -iP_0^{vac}\left(\delta x^+ + \frac{\bar{\beta}-\alpha}{\beta}\delta x^-\right) + \left(-i\frac{\alpha}{\pi}P_0^{vac} - \frac{2(n+1)L_0^{vac}}{n}\right)\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi\delta x^-}{\beta}\right).$$

- The Rényi mutual information [B. Chen, P. Hao, W. Song, in coming].

- The WCFT defined on a torus has modular properties [S. Detournay, T. Hartman, D. Hofman, 2012],
- A tours is defined by two identifications,

$$\begin{aligned} \textit{spatial circle}: \quad & (x^-, x^+) \sim (x^- + 2\pi, x^+)\,, \\ \textit{thermal circle}: \quad & (x^-, x^+) \sim (x^- + i\beta, x^+ - i\bar{\beta})\,, \end{aligned}$$

  where β and $\bar{\beta}$ are the inverse temperatures along x^- and x^+, respectively. The torus partition function can be written as

$$Z(\beta, \bar{\beta}) = \mathrm{Tr}(e^{-\beta L_0 + \bar{\beta} P_0})\,, \tag{11}$$

- The modular transformation that exchange two circles can be found,

$$x'^- = -i\frac{2\pi}{\beta}x^-, \quad x'^+ = x^+ + \frac{\bar{\beta}}{\beta}x^-\,. \tag{12}$$

- The partition function transforms according to the following equation with anomaly,

$$Z(\beta, \bar{\beta}) = e^{k\frac{\bar{\beta}^2}{4\beta}} Z\left(\frac{4\pi^2}{\beta}, -\frac{2\pi i\bar{\beta}}{\beta}\right)\,. \tag{13}$$

- Using the covariance of the partition, a Cardy-like formula for the asymptotic entropy has been found [S. Detournay, T. Hartman, D. Hofman, 2012],

$$S_{WCFT} = -\frac{4\pi i P_0 P_0^{vac}}{k} + 4\pi\sqrt{-\left(L_0^{vac} - \frac{(P_0^{vac})^2}{k}\right)\left(L_0 - \frac{P_0^2}{k}\right)}. \quad (14)$$

This formula reproduce the entropy of the WAdS black holes in TMG.

Structure constants of WCFT

- Make one step further, we can use the modular properties to determine the asymptotic behavior of one point function.
- For a primary operator $\mathcal{O}$ with conformal weight $h_{\mathcal{O}}$ and zero charge, the one point function on a torus is defined by

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = \mathrm{Tr}(\mathcal{O} e^{-\beta L_0 + \bar{\beta} P_0}) \,. \tag{15}$$

- Under the modular transformation the one point function transforms as

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = e^{k \frac{\bar{\beta}^2}{4\beta}} \left(\frac{\partial x'}{\partial x} \right)^{h_{\mathcal{O}}} \langle \mathcal{O} \rangle_{\frac{4\pi^2}{\beta}, -\frac{2\pi i \bar{\beta}}{\beta}} \,. \tag{16}$$

- Now take limit $\beta \to 0^+$, suppose the eigenvalues of L_0 are bounded from below, we have

$$\langle \mathcal{O} \rangle_{\beta,\bar{\beta}} = \langle \chi | \mathcal{O} | \chi \rangle \left(-i \frac{2\pi}{\beta} \right)^{h_{\mathcal{O}}} e^{-\frac{4\pi^2}{\beta} \Delta_\chi - \frac{2\pi i \bar{\beta}}{\beta} Q_\chi + k \frac{\bar{\beta}^2}{4\beta}} \tag{17}$$

where $|\chi\rangle$ is the lightest state with non-vanishing three-point co-

- The density of states weighted by the one point function can be written as an integral,

$$T_{\mathcal{O}}(\Delta, Q) = \int \frac{d\beta}{2\pi}\frac{d\bar{\beta}}{2\pi}\langle\mathcal{O}\rangle_{\beta,\bar{\beta}} e^{\beta\Delta-\bar{\beta}Q}$$

$$= \int \frac{d\beta}{2\pi}\frac{d\bar{\beta}}{2\pi}\langle\chi|\mathcal{O}|\chi\rangle \left(-i\frac{2\pi}{\beta}\right)^{h_{\mathcal{O}}} e^{-\frac{4\pi^2}{\beta}\Delta_\chi-\frac{2\pi i\bar{\beta}}{\beta}Q_\chi+k\frac{\bar{\beta}^2}{4\beta}+\beta\Delta-\bar{\beta}Q}.$$

- At large Δ and $-Q$, this integral is dominated by a saddle point with

$$\beta = 2\pi\sqrt{-\frac{\Delta_\chi^{inv}}{\Delta^{inv}}}, \quad \bar{\beta} = \frac{4\pi}{k}\left(iQ_\chi + Q\sqrt{-\frac{\Delta_\chi^{inv}}{\Delta^{inv}}}\right).$$

where $\Delta^{inv} = \Delta - \frac{Q^2}{k}$, $\Delta_\chi^{inv} = \Delta_\chi - \frac{Q_\chi^2}{k}$. When k is negative, the condition for the validity of the saddle point method is $\Delta^{inv} \gg 1$.

- $T_{\mathcal{O}}(\Delta, Q)$ characterise the total contribution of different degenerate states to the three-point function coefficient or the structure constant of the underling WCFT at given large Δ and $-Q$. However, it is useful to define the typical value of the three-point coefficient by dividing $T_{\mathcal{O}}(\Delta, Q)$ by the density of states,

$$\mathcal{C}_{\mathcal{O}}(\Delta, Q) \equiv \frac{T_{\mathcal{O}}(\Delta, Q)}{\rho(\Delta, Q)}\,. \tag{18}$$

Under saddle point approximation,

$$\mathcal{C}_{\mathcal{O}}(\Delta, Q) \sim (-i)^{h_{\mathcal{O}}} \langle \chi | \mathcal{O} | \chi \rangle \sqrt{\frac{\Delta_{\chi}^{inv}}{\Delta_{vac}^{inv}}} \sqrt{-\frac{\Delta^{inv}}{\Delta_{\chi}^{inv}}}^{h_{\mathcal{O}}} e^{4\pi\left(\sqrt{\frac{-\Delta_{\chi}^{inv}}{-\Delta_{vac}^{inv}}}-1\right)\sqrt{-\Delta_{vac}^{inv}\Delta^{inv}}-\frac{4\pi i}{k}\left(\frac{Q_{\chi}}{Q_{vac}}-1\right)Q_{vac}Q}\,.$$

- In 2D CFT, the asymptotic average structure constants calculated by the saddle point approximation has been down, and it has a geodesic one-loop interpretation in AdS_3 by considering a tadpole diagram. This work gives CFT evidence that the black hole geometry emerges upon course graining over microstates. [P. Kraus, A. Maloney, 2016].
- A more careful geodesic witten diagram in the bulk AdS_3 has been considered in order to reproduce the contribution from all descendent states of $|\chi\rangle$ [P. Kraus, A. Maloney, H. Maxfield, G. S. Ng, J. q. Wu, 2017].

- Under CSS boundary conditions, asymptotically AdS_3 spacetimes dual to WCFT.
- The BTZ metric in light-like coordinate,

$$ds^2 = \ell^2 \left(T_u^2 du^2 + 2\rho du dv + T_v^2 dv^2 + \frac{d\rho^2}{4(\rho^2 - T_u^2 T_v^2)} \right) , \tag{19}$$

$$u \sim u + 2\pi, \quad v \sim v + 2\pi \, .$$

The CSS boundary conditions,

$$g_{uv}^{(0)} = 1, \quad g_{vv}^{(0)} = 0, \quad \partial_v g_{uu}^{(0)} = 0, \quad g_{vv}^{(2)} = T_v^2 \, .$$

The asymptotic algebra,

$$\begin{aligned}
[\tilde{L}_n, \tilde{L}_m] &= (n-m)\tilde{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}, \\
[\tilde{L}_n, \tilde{P}_m] &= -m\tilde{P}_{n+m} + m\tilde{P}_0\delta_{n,-m}, \\
[\tilde{P}_n, \tilde{P}_m] &= \frac{\tilde{k}}{2} n\delta_{n,-m} \, ,
\end{aligned} \tag{20}$$

where

$$c = \frac{3\ell}{2G}, \quad \tilde{k} = -\frac{\ell T_v^2}{G} \, . \tag{21}$$

- $\tilde{L}_0$ and $\tilde{P}_0$ are the conserved charges associate with Killing vectors ∂_u and ∂_v respectively, which can be calculated,

$$\tilde{L}_0 = Q[\partial_u] = \frac{\ell T_u^2}{4G}, \quad \tilde{P}_0 = Q[\partial_v] = -\frac{\ell T_v^2}{4G}. \tag{22}$$

- This algebra however is different from the canonical algebra by a charge redefinition,

$$\tilde{L}_n = L_n - \frac{2P_0P_n}{k} + \frac{P_0^2\delta_{n,0}}{k}, \quad \tilde{P}_n = \frac{2P_0P_n}{k} - \frac{P_0^2\delta_{n,0}}{k}. \tag{23}$$

- This map can also be used to relate the energy and angular momentum in the bulk to the conformal dimensions and charges in the WCFT.

$$E = \tilde{L}_0 - \tilde{P}_0 = L_0 - 2\frac{P_0^2}{k}, \quad J = \tilde{L}_0 + \tilde{P}_0 = L_0 \tag{24}$$

- Using this map, the mean structure constant in WCFT can be rewritten as

$$\mathcal{C}_{\mathcal{O}}(\Delta, Q) = \mathcal{N} \underbrace{\langle\chi|\mathcal{O}|\chi\rangle}_{\text{cubic coupling}} \underbrace{T_u^{h_{\mathcal{O}}}}_{\text{from } \mathcal{O}} \underbrace{e^{\frac{\pi\ell}{2G}\left(\sqrt{\frac{-\Delta_\chi^{inv}}{-\Delta_{vac}^{inv}}}-1\right)T_u+\frac{\pi\ell}{2G}\left(\frac{Q_\chi}{Q_{vac}}-1\right)T_v}}_{\text{from } |\chi\rangle}, \tag{25}$$

where the normalization factor

$$\mathcal{N} = (-i)^{h_{\mathcal{O}}}\frac{1}{2}\left(\frac{\ell}{-4G\Delta_\chi^{inv}}\right)^{\frac{h_{\mathcal{O}}-1}{2}} \tag{26}$$

is independent of Δ and Q.

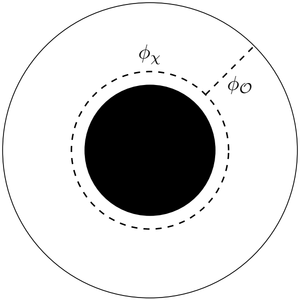


Figure: Configuration of trajectory for the holographic calculation of the average heavy-heavy-light three-point coefficient in the constant time slice of the BTZ black hole background.

- We work in the classical limit, and the propagator is given by e^{-S}, where S is the on-shell worldline action of a spinning particle,
- We consider heavy-heavy-light case,

$$1 \ll h_{\mathcal{O}} \ll \frac{c}{24}, \quad 1 \ll \Delta_{\chi}^{inv} + \frac{c}{24} < \frac{c}{24}, \quad 1 \ll -\frac{Q_{\chi}^{2}}{k} + \frac{c}{24} < \frac{c}{24}$$

- The worldline action [A. Castro, S. Detournay, N. Iqbal, 2014],

$$S_{\mathcal{O}} = \int d\tau \left(m_{\mathcal{O}} \sqrt{g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} + s_{\mathcal{O}}\tilde{n} \cdot \nabla n \right) + S_{constraints} \,. \quad (27)$$

$S_{constraints}$ contains Lagrange multipliers which require that the two normalized vectors n and $\tilde{n}$ should be mutually orthogonal and perpendicular to the worldline, namely

$$n^2 = -1, \quad \tilde{n}^2 = 1, \quad n \cdot \tilde{n} = 0, \quad n \cdot \dot{X} = \tilde{n} \cdot \dot{X} = 0 \,.$$

- The equations of motion with respect to $X^\mu(\tau)$ are known as the Mathisson-Papapetrou-Dixon (MPD) equations,

$$\nabla[m_{\mathcal{O}}\dot{X}^\mu + \dot{X}^\nu \nabla s^\mu{}_\nu] = -\frac{1}{2}\dot{X}^\nu s^{\rho\sigma} R^\mu{}_{\nu\rho\sigma}\,, \tag{28}$$

where $s^{\mu\nu}$ is the spin tensor,

$$s^{\mu\nu} = s_{\mathcal{O}}(n^\mu \tilde{n}^\nu - \tilde{n}^\mu n^\nu)\,. \tag{29}$$

- In locally AdS spacetimes, the contraction of the Riemann tensor with $s^{\mu\nu}\dot{X}^\rho$ vanishes. The MPD equations reduce to

$$\nabla[m_{\mathcal{O}}\dot{X}^\mu - s^\mu{}_\nu \nabla \dot{X}^\nu] = 0\,. \tag{30}$$

One obvious solution to the MPD equation above is a geodesic

$$\nabla \dot{X}^\mu = 0\,. \tag{31}$$

- As discussed in [A. Castro, S. Detournay, N. Iqbal, 2014], the spin's contribution can be written as

$$S_{\text{spin}} = s_{\mathcal{O}} \log\left(\frac{q(\tau_f)\cdot n_f - \tilde{q}(\tau_f)\cdot n_f}{q(\tau_i)\cdot n_i - \tilde{q}(\tau_i)\cdot n_i}\right), \tag{32}$$

where two vectors q^μ and $\tilde{q}^\mu$ are mutually orthogonal, perpendicular to the geodesic, and furthermore are parallel transported along and the geodesic, i.e.,

$$q^2 = -1, \quad \tilde{q}^2 = 1, \quad q\cdot\tilde{q} = 0, \quad q\cdot\dot{X} = \tilde{q}\cdot\dot{X} = 0, \quad \nabla q = \nabla\tilde{q} = 0.$$

- The two sets of vectors $(n(\tau), \tilde{n}(\tau))$ and $(q(\tau), \tilde{q}(\tau))$ can be related via a Lorentz boost. In fact, we can expand $n(\tau)$ and $\tilde{n}(\tau)$ in terms of $q(\tau)$ and $\tilde{q}(\tau)$,

$$n(\tau) = \cosh(\eta(\tau))q(\tau) + \sinh(\eta(\tau))\tilde{q}(\tau), \tag{33}$$

$$\tilde{n}(\tau) = \sinh(\eta(\tau))q(\tau) + \cosh(\eta(\tau))\tilde{q}(\tau), \tag{34}$$

where $\eta(\tau)$ is the rapidity of this Lorentz boost.

- The spin part count the difference in the rapidity between initial and final point,

$$S_{\text{spin}} = s_{\mathcal{O}}(\eta(\tau_f) - \eta(\tau_i))\,. \tag{35}$$

- Wrap them up, we can write down the on-shell action for $\phi_{\mathcal{O}}$ and ϕ_{χ} on the BTZ background,

$$S_{\mathcal{O}} = -\log\left(T_u^{\frac{\ell m_{\mathcal{O}}+s_{\mathcal{O}}}{2}} T_v^{\frac{\ell m_{\mathcal{O}}-s_{\mathcal{O}}}{2}}\right)\,. \tag{36}$$

$$S_{\chi} = \ell m_{\chi} 2\pi(T_v + T_u) + s_{\chi} 2\pi(T_u - T_v)\,. \tag{37}$$

- The non-perturbative particle ϕ_χ will backreact to the background geometry to give a rotational conical defect, This is the result of non-vanishing localized energy and angular momentum source appeared in three dimensional Einstein equations.

$$\begin{aligned} ds^2 &= \ell^2 \left[-(1+r^2) \left(dt - \frac{s_\chi}{\frac{\ell}{4G}\left(1-\frac{\delta\varphi}{2\pi}\right)} d\varphi \right)^2 + \frac{dr^2}{1+r^2} + r^2 d\varphi^2 \right], \qquad (38) \\ \varphi &\sim \varphi + 2\pi - \delta\varphi\,. \qquad (39) \end{aligned}$$

Here the deficit angle $\delta\varphi$ is related to the mass of the ϕ_χ through

$$m_\chi = \frac{\delta\varphi}{8\pi G}\,.$$

- To see the relation the mass and spin to its boundary quantum numbers, we put the conical defect solution into standard light-like BTZ form,

$$ds^2 = \ell^2 \left[T_{\chi u}^2 du^2 + 2\rho du dv + T_{\chi v}^2 dv^2 + \frac{d\rho^2}{4(\rho^2 - T_{\chi u}^2 T_{\chi v}^2)} \right], \tag{40}$$

$$u \sim u + 2\pi, \quad v \sim v + 2\pi \tag{41}$$

where

$$T_{\chi u}^2 = -\frac{(1 - \delta\varphi/2\pi - 4Gs_\chi/\ell)^2}{4}, \quad T_{\chi v}^2 = -\frac{(1 - \delta\varphi/2\pi + 4Gs_\chi/\ell)^2}{4}.$$

- Using the map, we found,

$$m_\chi = \frac{1}{4G} - \frac{1}{8G}\left(\sqrt{\frac{-\Delta_\chi^{inv}}{-\Delta_{vac}^{inv}}} + \frac{Q_\chi}{Q_{vac}}\right), \quad s_\chi = -\frac{\ell}{8G}\left(\sqrt{\frac{-\Delta_\chi^{inv}}{-\Delta_{vac}^{inv}}} - \frac{Q_\chi}{Q_{vac}}\right). \tag{42}$$

The total amplitude for the process given by the one loop tadpole diagram under CSS boundary conditions,

$$\begin{aligned} \mathcal{C}^{bk}_{\mathcal{O}}(\Delta, Q) &= \underbrace{\langle\chi|\mathcal{O}|\chi\rangle}_{\text{vertex}} e^{-S_{\mathcal{O}}} e^{-S_{\chi}} \\ &= \underbrace{\langle\chi|\mathcal{O}|\chi\rangle}_{\text{vertex}} \underbrace{T_u^{h_{\mathcal{O}}}}_{\langle\phi_{\mathcal{O}}\phi_{\mathcal{O}}\rangle} \underbrace{e^{\frac{\pi\ell}{2G}\left(\sqrt{\frac{-\Delta^{inv}_{\chi}}{-\Delta^{inv}_{vac}}}-1\right)T_u+\frac{\pi\ell}{2G}\left(\frac{Q_{\chi}}{Q_{vac}}-1\right)T_v}}_{\langle\phi_{\chi}\phi_{\chi}\rangle}, \end{aligned}$$

Summary and outlook

- WCFT is a 2d quantum field theory with warped conformal field theory, arise in the study of dual field theory of WAdS.
- Similar to 2d CFT, the correlation functions, density of state, and entanglement entropy of WCFT can be determined by the symmetry.
- Using the modular properties, the asymptotic structure constants can be determined, and we find its gravity dual in AdS_3 with CSS boundary.
- Our result indicates that the black hole geometries in asymptotically AdS_3 spacetimes can emerge upon course graining over microstates in WCFTs.
- The contribution from the descendent states of $|\chi\rangle$ to the average structure constants? The geodesic Witten diagram approach in $(W)AdS_3$?

Thank You!