# On Noether's Theorem and Gauge-Gravity Duality

Sebastian De Haro

University of Amsterdam and Harvard University

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# Outline

# Introduction

Noether's theorem applied to general relativity implies that the conservation law for the energy and momentum of the gravitational field is an identity (the Bianchi identity), rather than requiring satisfaction of the equations of motion of GR.

Klein and Hilbert took this to mean that there is no analogue of energy conservation in general relativity.

Research after Penrose (1982) shifted to defining *quasilocal* quantities: in particular, to defining suitable tensors over spacelike surfaces that reproduce well-known formulas such as the ADM mass for various solutions of Einstein's equations.

Tensors thus defined still have some ambiguities, due to the possibility to add boundary terms without changing the equations of motion.

In AdS-CFT, the asymptotic Brown-York (1993) stress-energy tensor is identified with the expectation value of the stress-energy tensor of the CFT.

# Noether's Theorems and the Energy-Momentum Tensor

Consider Maxwell's theory (with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$):

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}\, F^{\mu\nu} \;, \quad E_\mu(A) = \partial^\nu F_{\mu\nu} = 0 \;.$$

**Noether's first theorem:** the conservation of the energy-momentum tensor density $T_{\mu\nu}$ follows from the translational symmetry of Minkowski spacetime, i.e. under $\delta x^\mu = \xi^\mu$:

$$\begin{aligned} T^\mu{}_\nu &:= \delta^\mu_\nu\, \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\, \partial_\nu A_\lambda = F^{\mu\lambda}\, \partial_\nu A_\lambda - \frac{1}{4}\, \delta^\mu_\nu\, F^{\lambda\sigma}\, F_{\lambda\sigma} \\ \partial_\mu T^\mu{}_\nu &= \text{(zero by e.o.m.)} \qquad\qquad \text{(weak conservation law)} \end{aligned}$$

The energy-momentum tensor admits "improvement terms" (Belinfante 1939):

$$T'^\mu{}_\nu := T^\mu{}_\nu + \partial_\lambda\, U^{[\mu\lambda]}{}_\nu$$

is also conserved. Such improvement terms are *one way* to bring define an energy-momentum tensor with the standard form:

$$T'^\mu{}_\nu = F^{\mu\lambda}\, F_{\nu\lambda} - \frac{1}{4}\, \delta^\mu_\nu\, F^{\lambda\sigma}\, F_{\lambda\sigma} + \text{(zero by eom)} \;.$$

# Noether's Theorems and the Energy-Momentum Tensor

General relativity, through its equation $G_{\mu\nu} = -8\pi G_{\rm N}\, T_{\mu\nu}$, *uniquely* detects the energy-momentum tensor of its sources—it fixes the "improvement term". The one that appears when coupling GR to Maxwell's theory is indeed the standard one.

**Noether's second theorem:** the invariance of Maxwell's action under $\delta A^\mu = \partial^\mu\lambda$ implies $\partial^\mu E_\mu(A) = \partial^\mu\partial^\nu F_{\mu\nu} = 0$ identically, by the anti-symmetry of $F_{\mu\nu}$. (If we couple Maxwell's theory to a source $J_\mu$, then we find $\partial^\mu J_\mu = 0$, i.e. charge conservation. Thus a theory with a gauge symmetry can only be coupled to a conserved current.)

General relativity in vacuum: $E_{\mu\nu}(g) = G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0$. The invariance of the Einstein-Hilbert action under an infinitesimal coordinate transformation, $\delta x^\mu = \xi^\mu(x)$, $\delta g_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$, leads to the 4 (contracted) **Bianchi identities:** $\nabla^\mu G_{\mu\nu} = 0$.

If the theory is coupled to matter, this gives the covariant conservation law $\nabla^\mu T_{\mu\nu} = 0$.

# Noether's Theorems (Brading and Brown, 2003)

**Noether's first theorem** considers transformations with $\rho$ *constant parameters*, $\eta$, i.e. global symmetries. For every continuous global symmetry there exists a conservation law (and viceversa):

$$\left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\right)\eta^i_k = \partial_\mu j^\mu_k \ , \quad k = 1, \cdots, \rho \ .$$

The $\rho$ currents $j^\mu_k$ are conserved provided the Euler-Lagrange equations are satisfied. Hilbert and Noether called these **proper conservation laws**, because they are not identities, i.e. they are non-trivially satisfied. (The theorem also gives an independent expression for $j^\mu_k$).

**Noether's second theorem** gives two equations: a set of identities between the equations of motion, and a conservation law:

$$\partial_\mu\left(j^\mu_k - \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\nu \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i)}\right) b^\mu_{ki}\right) = 0 \qquad (1)$$

# Noether's Theorems (Brading and Brown, 2003)

Hilbert and Noether called the latter an **improper conservation law**, because it does not require use of the equations of motion. One can define a new current, $\Theta$, as follows:

$$\begin{aligned} \Theta^\mu_k &:= j^\mu_k - \left(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\nu\,\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_i)}\right) b^\mu_{ki}\,, \qquad \partial_\mu\,\Theta^\mu_k = 0 \\ \Rightarrow \exists\, U:\quad \Theta^\mu_k &= \partial_\nu\, U^{\mu\nu}_k\,, \qquad\qquad U^{\mu\nu}_k = -U^{\nu\mu}_k \\ \partial_\mu\,\partial_\nu\, U^{\mu\nu}_k &= 0\,. \end{aligned}$$

$U$ is called the **superpotential.** The conservation of $\Theta$ is an identity (Bianchi identity), and it follows from $U$'s antisymmetry (recall Maxwell's theory, where under a gauge transformation: $\partial_\nu F^{\mu\nu} = \partial_\mu J^\mu = 0$).

# Noether's Theorems

**Noether's "third theorem":** 'Given [an action] $S$ invariant under the group of translations, then the energy relations [i.e. the conservation laws corresponding to translations] are improper [i.e. the divergences vanish identically] iff $S$ is invariant under an infinite group which contains the group of translations as a subgroup.' (Noether, 1918).

Thus only if the finite group is a subgroup of an infinite dimensional group are the conservation laws of the finite group improper (i.e. they are identities). Otherwise they are always proper, i.e. non-trivial.

So far so good. Let us now look at Klein and Hilbert's interpretation of the theorem—and their worries.

# Klein's and Hilbert's worries

Zu Hilberts erster Note über die Grundlagen der Physik [1]).

Von

**F. Klein.**

Vorgelegt in der Sitzung vom 25. Januar 1918.

**I. Aus einem Briefe von F. Klein an D. Hilbert.**

... Indem ich Ihre Note sorgfältig studierte, bemerkte ich, daß man die Zwischenrechnungen, die Sie anstellen, durch Benutzung des gewöhnlichen Lagrangeschen Variationsansatzes

Klein to Hilbert: 'You know that Frl. Noether advises me continuously in my work, and that it is actually only through her that I have penetrated in this matter. When I recently told Frl. Noether about my results on your energy vector, she announced that she had derived the same consequences from your note already a year ago, and that she wrote it down in a manuscript back then.' [Klein communicated two of Noether's papers to the Königlichen Gesellschaft der Wissenschaften zu Göttingen.]

# Klein's and Hilbert's worries

**Felix Klein** (1917) worried that, in GR, energy and momentum conservation are *identities*, without requiring the equations of motion to be satisfied (they are improper). It is a *strong conservation law.*

**David Hilbert:** in general relativity, there are no proper conservation laws (Kosmann-Schwarzbach, p. 63).

Hilbert, 1917: 'I hoped that, in the case of the general theory of relativity, energy equations that correspond to the energy equations of [classical mechanics]... do not exist at all. Indeed I would like to denote this circumstance as a distinctive feature of the general theory of relativity'.

Noether's "third theorem" proves that in GR there is no proper description of energy and momentum (of this type): 'As Hilbert expresses his assertion, the lack of a proper energy law constitutes a characteristic of the general theory of relativity'.

# Klein's challenge to Einstein

statt hat, erscheint als mathematische Identität. Besagte Angabe kann also wohl nicht als Analogie zum Erhaltungssatze der Energie, wie er in der gewöhnlichen Mechanik herrscht, angesehen werden. Denn wenn wir in letzterer schreiben:

$$\frac{d(T+U)}{dt} = 0,$$

so besteht diese Differentialbeziehung doch nicht identisch, sondern erst in Folge der Differentialgleichungen der Mechanik.

'Thus the above statement cannot be regarded analogous to the conservation law of energy in ordinary mechanics. For when we write the latter as $d(T+U)/dt = 0$, this differential equation is not satisfied identically [as the Bianchi identity is], but only as a consequence of the differential equations of mechanics.'

Klein (1917): 'After all this, I can barely believe that it is useful to designate [Einstein's] very arbitrarily built quantities $t^{\mu}{}_{\nu}$ [the pseudotensor] as the components of the energy of the gravitational field.'

# Klein's and Hilbert's worries

However, as the literature has emphasised, from the fact that the conservation law is improper, it does not by itself follow that the law could not express a physical (non-)conservation law.
(A more basic question is: What is the definition of energy and momentum for the gravitational field?)

A case in point is Maxwell's theory: While in vacuum, gauge invariance implies $\partial_\mu \partial_\nu F^{\mu\nu} = 0$, which is trivially satisfied by the antisymmetry of the Faraday tensor: when we couple Maxwell's theory to matter, gauge invariance leads to **charge conservation:**

Thus *Noether's second theorem restricts the properties of the matter that can be coupled to pure Maxwell's theory while preserving its gauge symmetry,* and the ensuing conservation law does have physical content.

But even taking this in our stride: if we try to assign a physical significance to the conservation of the energy-momentum tensor in GR along similar lines, **two challenges lie ahead...**

# Klein's challenge and Einstein's proposal

Two further questions that Klein's and Hilbert's challenge suggest:

(i) When GR is coupled to matter, the Bianchi identity implies the conservation law $\nabla_\mu T^\mu{}_\nu = 0$. But $T^\mu{}_\nu$ is the energy-momentum tensor of **matter rather than the gravitational field.** So, what is the definition of the energy and momentum *of the gravitational field itself*?

(ii) $\nabla_\mu T^\mu{}_\nu = 0$ is not a conservation law in the usual sense, for the energy-momentum tensor **cannot be straightforwardly integrated to give a conserved vector current** (this *can* be done if there is a Killing vector).

Einstein's idea: $\nabla_\mu T^\mu{}_\nu = 0$ contains a coupling to gravity through the Christoffel symbols. So, try to construct an energy-momentum tensor for *gravity* from the Christoffel symbols—viz. introduce a **pseudotensor.**

# Einstein's Pseudotensor

Einstein (1916) introduced a *pseudotensor* $t^\mu{}_\nu$ such that the conservation law contains an *ordinary derivative*:

$$\partial_\mu \left(\sqrt{g}\, T^\mu{}_\nu + t^\mu{}_\nu\right) = 0 \quad \Rightarrow \quad \nabla_\mu \, T^\mu{}_\nu = 0 \; .$$

He interpreted $t^\mu{}_\nu$ as a "gravitational energy-momentum tensor" density. One can associate with this tensor a *superpotential*, $s^{\mu\lambda}{}_\nu$, such that:

$$\sqrt{g}\, T^\mu{}_\nu + t^\mu{}_\nu = \partial_\lambda \, s^{\mu\lambda}{}_\nu \quad , \quad \partial_\mu \, \partial_\lambda \, s^{\mu\lambda}{}_\nu = 0 \; .$$

Einstein obtained the superpotential from a boundary term in the action (his 1916 action differed from the Einstein-Hilbert action by a boundary term).

Problem: $t$ is **not a tensor!** And the gravitational energy density, $t$, can be made to vanish at any given point in appropriately chosen coordinates.

# Other Pseudotensors

There are an infinite number of pseudotensors. For define a pseudotensor with an arbitrary superpotential:

$$\begin{aligned} 16\pi G_{\rm N}\, t^{\mu}{}_{\nu} &:= -2\sqrt{g}\, G^{\mu}{}_{\nu} + \partial_\lambda\, U^{\mu\lambda}{}_{\nu} \\ \Rightarrow\ 16\pi G_{\rm N}\left(t^{\mu}{}_{\nu} + \sqrt{g}\, T^{\mu}{}_{\nu}\right) &= \partial_\lambda\, U^{\mu\lambda}{}_{\nu} \end{aligned}$$

still satisfies the conservation laws as long as $U$ is antisymmetric:

$$U^{\mu\lambda}{}_{\nu} = U^{[\mu\lambda]}{}_{\nu}\ .$$

Challenges:

(i) The pesudotensor is not covariant.
(ii) There are an infinite number of them.

# Quasi-Local Energy: Penrose (1982)

Schrödinger (1950) called the pseudotensors 'sham tensors'.

Misner, Thorne, Wheeler (1973, p. 467): 'Anyone who looks for a magic formula for "local gravitational energy-momentum" is looking for the right answer to the wrong question. Unhappily, **enormous time and effort were devoted in the past to trying to "answer this question"** before investigators realized the **futility of the enterprise.'**

Clearly, attempts to define *local* energy and momentum failed, except in special circumstances (spacetimes with Killing vectors).

Penrose (1982): one can still define quasi-local energy and momentum, i.e. associated with a closed spacelike surface. It refers only to the geometry of the 2-surface and the extrinsic curvature quantities for its embedding in the spacetime.

# A Rehabilitation of Pseudotensors? Nester et al. ('98, '18)

Chen, Nester and collaborators (1998, 2015, 2018) have defended a rehabilitation of pseudotensors. They argue that pseudotensors:

(a) Do provide a description of energy and momentum conservation.
(b) Have well-defined values in each system of coordinates.
(c) Give at infinity the expected values of the total energy and momentum.

I will look at (c) more closely.

# Chen, Nester et al. (1998, 2018)

The superpotential, ${U^{\mu\lambda}}_{\nu}$, determines the boundary term as a 2-surface integral that contributes to the Hamiltonian:

$$H(N) = \int_\Sigma N^\mu \mathcal{H}_\mu + \oint_{\partial\Sigma} \mathcal{B}(N) \, , \qquad \mathcal{B}(N) = \frac{1}{2} N^\lambda \, {U^{\mu\nu}}_\lambda \, \mathrm{d}\Sigma_{\mu\nu}$$

where $\Sigma$ is a spacelike hypersurface and $N^\mu$ is a vector field that generates displacements along $\Sigma$.

Noether's theorem implies that $\mathcal{H}_\mu$ is proportional to the equations of motion, and so the Hamiltonian reduces to a **boundary term**, which is given by the superpotential. And so, the pseudotensor gives the quasilocal energy (it depends only on the choice of coordinates on $\partial\Sigma$ not $\Sigma$).

Chen, Nester et al. argue that different choices of pseudotensors in the literature correspond to different such boundary terms. The latter specify the variational problem, i.e. which fields are held fixed at the boundary. Regardless of controversies about pseudotensors: *the boundary term determines the boundary conditions and the quasilocal energy.*

# Brown and York's (1993) Quasilocal Stress-Energy Tensor

Brown and York (1993) use an analogue of the Hamilton-Jacobi method from classical mechanics: they define energy and momentum as the conjugates of the classical action. Up to terms proportional to the equations of motion, these are given by boundary terms.

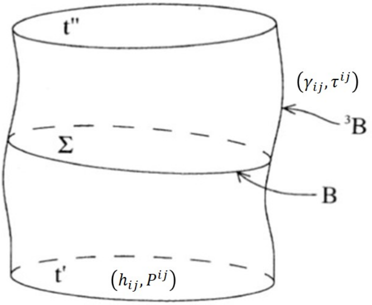



FIG. 1. Spacetime $M$ with boundary $\partial M$, which consists of initial and final spacelike hypersurfaces $t'$ and $t''$ and a timelike three-surface $^3B$. A generic spacelike slice $\Sigma$ has two-boundary $B$.

# Brown and York's (1993) Quasilocal Stress-Energy Tensor

$$S=\frac{1}{2\kappa}\int_M \mathrm{d}^4x\,\sqrt{-g}\;R\;\underbrace{+\frac{1}{\kappa}\int_{t'}^{t''}\mathrm{d}^3x\,\sqrt{h}\;K}_{\text{extrinsic curvature of hypersurface }\Sigma}\;\underbrace{-\frac{1}{\kappa}\int_{^3B}\sqrt{-\gamma}\;\Theta}_{\text{extrinsic curvature of }^3B}+S^{\mathrm{ref}}+S^{\mathrm{matter}}$$

$$\delta S=(\mathrm{eom})+\int_{t'}^{t''}\mathrm{d}^3x\;P^{ij}\,\delta h_{ij}+\int_{^3B}\mathrm{d}^3x\;\tau^{ij}\,\delta\gamma_{ij}+\text{(matter bdy terms)}\;.$$

The reference term subtracts the same expression, "on a reference spacetime" (e.g. Minkowski space). It is needed to make the expressions finite (more on this later). The result is:

$$\tau^{ij}=-\frac{1}{\kappa}\left(\Theta\,\gamma^{ij}-\Theta^{ij}\right)+\text{(subtracted terms)}\;.$$

The boundary stress-energy tensor satisfies one of Einstein's equations: $\nabla_i\,\tau^{ij}=-T^{nj}$, where $T^{nj}$ is the energy-momentum tensor with one index projected normally, and the other tangentially, to $^3B$. Thus there is a contribution from the matter that passes through the boundary.
If the boundary three-metric $\gamma_{ij}$ possesses a Killing vector field $\xi_i$, then $\tau^{ij}$ defines a conserved charge, which is conserved if $T^{nj}=0$.

# AdS-CFT

# A Brief Introduction to AdS-CFT ('holography')

Gauge-gravity duality is a conjectured duality (an isomorphism) between:
(i) String theory in a $(d+1)$-dimensional spacetime that is asymptotically locally anti-de Sitter (AdS). At low energies: supergravity.
(ii) A quantum field theory (usually, conformal: CFT: no gravity!) on the conformal boundary of this manifold, i.e. in $d$ dimensions. The CFT is strongly coupled if (i) is weakly coupled.

The duality is, roughly, as follows: to each field, of mass $m$, in the gravity theory corresponds an operator $\mathcal{O}_\Delta(x)$ of dimension $\Delta(m)$ in the CFT (e.g. $\mathcal{O}_{\Delta=4} = \mathrm{Tr}\, F^2$ in supersymmetric Yang-Mills theory).

As we approach the boundary, fields in AdS typically have a *leading non-normalizable mode* and a *sub-leading, normalizable, mode*. The value of the **normalizable** mode corresponds to the expectation value, $\langle\mathcal{O}_\Delta(x)\rangle$, of the operator in a particular boundary state (detemined by the leading value of the field). The value of the **non-normalizable** mode coresponds to a boundary *source*, $J(x)$, that couples to the operator in the Lagrangian, i.e. by a term $\int \mathrm{d}^d x\, J(x)\, \mathcal{O}(x)$.

# A Brief Introduction to AdS-CFT (Maldacena 1997, Witten 1998)

At the semi-classical level, the AdS-CFT correspondence can be summarised by the following statement (in Euclidean signature):

$$\exp\left(-S^{\text{on-shell}}_{\text{supergravity}}[\phi(r,x)]\right)\Big|_{r^{\Delta-d}\,\phi(r,x)|_{r=0}=J(x)} = \left\langle e^{\int \mathrm{d}^d x\ J(x)\,\mathcal{O}(x)}\right\rangle$$
$$=: \quad Z_{\text{CFT}}[J] =: exp\left(-W_{\text{CFT}}[J]\right)$$

(Here, $\phi(r,x)$ stands for a (collection of) scalar field(s), and $r=0$ is the boundary. The correspondence holds for vector and tensor fields as well.)

**Correlation functions** of arbitrary numbers of operators $\langle\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle$ can be calculated by taking functional derivatives of the partition function.

One- and two-point functions: the *classical* gravitational action suffices, and there is complete agreement with the CFT.

Higher-point functions: need to take into account quantum (string theory) corrections, which appear as higher-curvature (higher-derivative) terms in the effective action.

# Example: a 2-point function

Consider a conformally coupled scalar field in $AdS_4$, i.e. $d = 3$. Solve the equations of motion by expanding asymptotically, i.e. near $r = 0$:

$$\Phi(r, x) \;=\; r\,\phi_{(0)}(x) + r^2\,\phi_{(1)}(x) + \ldots$$

Requiring that the solution is regular at $r = \infty$ gives:

$$\begin{aligned}\Phi(r, x) &= \frac{1}{\pi^2}\int \mathrm{d}^3x' \, \frac{r^2}{(r^2 + (\mathbf{x}-\mathbf{x}')^2)^2}\,\phi_{(0)}(x) \\ &= r\,\phi_{(0)}(x) + \frac{r^2}{\pi^2}\int \mathrm{d}^3x' \, \frac{\phi_{(0)}(x')}{(\mathbf{x}-\mathbf{x}')^4} + \ldots\end{aligned}$$

$$\Rightarrow \quad \langle \mathcal{O}_{\Delta=2}(x)\,\mathcal{O}_{\Delta=2}(x')\rangle_{J=0} \;=\; \left.\frac{\delta^2 W_{\rm CFT}[J]}{\delta J(x)\,\delta J(x')}\right|_{J=0} = \frac{1}{\pi^2(\mathbf{x}-\mathbf{x}')^4}\;,$$

as expected for a scalar operator of dimension 2 in a CFT.

# The Dual Stress-Energy Tensor (De Haro et al., 2000; Balasubramanian Kraus, 1999; Larsen et al. '99)

In quantum field theories, the metric is usually fixed to, say, a flat metric. But we can consider quantum field theories on a fixed curved background metric $g_{ij(0)}(x)$, and regard $g_{ij(0)}(x)$ as a source for the stress-energy tensor. In this way the expectation value of the stress-energy tensor of the QFT is given by the functional derivative of the partition function:

$$\langle T_{ij}(x)\rangle = \frac{2}{\sqrt{g_{(0)}}}\,\frac{\delta W_{\text{CFT}}[g_{(0)}]}{\delta g^{ij}_{(0)}(x)}$$

**Gravity description:** $g_{ij(0)}$ is the asymptotic metric along the boundary directions, up to a conformal factor. A theorem of Fefferman and Graham (1985, 2012) guarantess that in an open neighbourhood of the boundary we can write the metric in **Poincaré normal form:**

$$\begin{aligned} \mathrm{d}s^2 = \frac{\ell^2}{r^2}\,\mathrm{d}r^2 + \gamma_{ij}(r,x)\,\mathrm{d}x^i\,\mathrm{d}x^j &= \frac{\ell^2}{r^2}\left(\mathrm{d}r^2 + g_{ij}(r,x)\,\mathrm{d}x^i\,\mathrm{d}x^j\right), \\ g_{ij(0)}(x) &:= g_{ij}(0,x)\ . \end{aligned}$$

# The Dual Stress-Energy Tensor (De Haro et al., 2000; Balasubramanian Kraus, 1999; Larsen et al. '99)

We can use the Brown-York approach to write the on-shell gravity action as a function of the arbitrary boundary metric, $g_{ij(0)}(x)$.
The duality then identifies the generating functional of the CFT with the on-shell gravity action, so that we get the following identification:

$$\langle T_{ij}(x)\rangle = \frac{2}{\sqrt{g_{(0)}}}\frac{\delta W_{\text{CFT}}[g_{(0)}]}{\delta g^{ij}_{(0)}(x)} \qquad \text{('holographic stress-energy tensor')}$$

$$\equiv \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\text{on-shell}}[g_{(0)}]}{\delta g^{ij}_{(0)}(x)} = \frac{d\,\ell^{d-1}}{16\pi G_{\text{N}}}\,g_{(d)ij}(x) + \text{(local terms)} = \tau_{ij}\ .$$

Thus the duality **identifies the quasi-local Brown-York stress-energy tensor with the stress-energy tensor of the CFT.** Here, $g_{(d)}(x)$ is the $d$-th term in the Taylor expansion of the metric $g_{ij}(r,x)$ around $r=0$.

This gives a **completely general procedure to calculate the stress-energy tensor:** take any solution, Taylor expand the metric around $r=0$, and calculate the $d$-th coefficient.

# Examples

Three-dimensional BTZ black hole (Balasubramanian and Kraus, 1999):

$$\langle T_{00}\rangle = \frac{M}{2\pi\ell}\ , \quad \langle T_{01}\rangle = -\frac{J}{2\pi\ell}$$

$AdS_4$-Schwarzschild black hole:

$$\langle T_{00}\rangle = \frac{R_{\rm S}}{8\pi G\,\ell} + \ldots \quad \Rightarrow \; M = \frac{R_{\rm S}}{2G_{\rm N}}\ .$$

4D solutions with self-dual Weyl tensor, $C_{\mu\nu\alpha\beta} = \frac{1}{2}\,\epsilon_{\mu\nu\gamma\delta}\,C_{\alpha\beta}{}^{\gamma\delta}$ (De Haro, 2009):

$$\langle T_{ij}\rangle = \frac{\ell^2}{8\pi G}\,C_{ij}[g_{(0)}] \;=\; \frac{\ell^2}{16\pi G_{\rm N}}\,\epsilon_i{}^{kl}\,\nabla_k^{(0)}\left(R_{jl}[g_{(0)}] - \frac{1}{4}\,g_{(0)jl}\,R[g_{(0)}]\right)$$

integrated to: $$W[\Gamma_{(0)}] \;=\; -\frac{1}{4}\int \mathrm{Tr}\left(\Gamma_{(0)}\wedge d\Gamma_{(0)} + \frac{2}{3}\,\Gamma_{(0)}\wedge\Gamma_{(0)}\wedge\Gamma_{(0)}\right).$$

# Boundary diffeomorphisms

In the presence of sources, the boundary stress-energy tensor is not conserved, but satisfies an identity that relates its covariant divergence to the expectation value of the operators that couple to the sources.
Consider the partition function:

$$Z_{\mathrm{CFT}}[g_{(0)},\phi_{(0)}]=\left\langle \exp\int \mathrm{d}^d x\,\sqrt{g_{(0)}}\left(\frac{1}{2}\,g^{ij}_{(0)}\,T_{ij}-\phi_{(0)}\,\mathcal{O}\right)\right\rangle\,.$$

Demand its invariance under infinitesimal changes of coordinates:

$$\delta g_{(0)ij}=\nabla_i\,\xi_j+\nabla_j\,\xi_i\ ,$$

which yields the (sometimes called 'Ward') identity:

$$\nabla^j\,\langle T_{ij}\rangle=\langle\mathcal{O}\rangle\,\partial_i\phi_{(0)}\ .$$

One can check that this result is reproduced, on the gravity side, by the identity of Brown and York (1993) for a scalar field coupled to gravity:

$$\nabla_j\,\tau^{ij}=-T^{ni}\ .$$

For Noether's theorem in connection with *boundary symmetries*, see Papadimitriou and Skenderis (2005).

# Holographic renormalization

**Renormalization:** the stress-energy tensor, if calculated directly from the supergravity action, is divergent (because of the infinite volume: recall the need for a "reference spacetime"). Thus in the previous examples we regularized the action imposing a cutoff $r = \epsilon$, and renormalized adding boundary counterterms to the action (afterwards taking $\lim_{\epsilon\to 0}$) that:

(i) Do not change the equations of motion.
(ii) Are local in the boundary coordinates. They are of the type: $\int \mathrm{d}^d x \sqrt{\gamma}$, $\int \mathrm{d}^d x \sqrt{\gamma}\, R[\gamma]$, and higher powers. Likewise for matter fields.
(iii) Render the on-shell action and the stress-energy tensor finite.

This way of renormalizing has a clearer interpretation than the old *subtraction method*: namely, as counterterms in the CFT.

The requirement of finiteness only fixes the IR divergent part of the effective action (counterterms). **Finite boundary terms** can still be added. If the Chen, Nester et al. results are correct, then the finite term ambiguity should be related to the possibility of **defining different pseudotensors.**

# Bulk diffeomorphisms

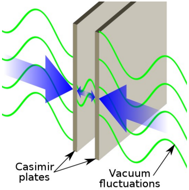
For **odd boundary dimension** $d$**,** the stress-energy tensor $\tau_{ij} \equiv \langle T_{ij} \rangle$ is *covariant* under bulk diffeomorphisms that preserve the Poincaré normal form of the metric.

But for **even** $d$**,** the renormalized Brown-York quasi-local stress-energy tensor transforms *anomalously* under **bulk diffeomorphisms that reduce to a conformal transformation** on the boundary.

This matches the **conformal anomaly** of the CFT on a non-flat background, and it implies that the stress-energy tensor is not traceless quantum mechanically. This is well-known, and it does *not* signal an inconsistency of the QFT on a curved background: for the conformal symmetry is *not* a defining symmetry of the theory—it just happens to be there in the classical theory. (It is an "accidental symmetry" of the classical limit of the theory').

# Casimir energy in CFT

In electrodynamics, the zero-point energy of the electromagnetic field between two conducting plates manifests itself as a force between the plates (the Casimir effect).




In CFT, a conformal transformation from a flat to a compact space (e.g. from $\mathbb{R}^2 \to S^2$) can have a similar effect. While the anomaly vanishes on $\mathbb{R}^2$, it is non-zero in $S^2$:

$$\mathcal{A} = -\frac{c}{24\pi}\, R\,, \qquad\qquad c \equiv \frac{3\ell}{2G_{\rm N}}\,.$$

And so, even though $\langle T_{00}(\mathbb{R}^2)\rangle = 0$, we find $\langle T_{00}(S^2)\rangle \sim c\,R$.

# Bulk diffeomorphisms

Consider diffeomorphisms that "fix the location of the boundary $r = 0$":

$$\delta x^i = \xi^i(x) \,, \quad \delta r = -r\,\xi(x) \,.$$

Requiring that these diffeomorphisms fix: (i) the Poincaré normal form of the metric and (ii) the boundary metric, implies:
(Brown and Henneaux (1986), De Haro (2017)):

$$\nabla_{(0)i}\,\xi_j(x) + \nabla_{(0)j}\,\xi_i(x) = \frac{2}{d}\,g_{(0)ij}(x)\,\nabla^k_{(0)}\,\xi_k \quad \text{and} \quad \xi(x) = -\frac{1}{d}\,\nabla^k\xi_k \,.$$

This is the **conformal Killing equation,** which is precisely the condition for $\xi^i(x)$ to be a conformal transformation. This is the well-known statement that 'the asymptotic symmetry group of AdS is the conformal group', now for an *arbitrary boundary metric*.

Henningson and Skenderis (1998) argue that the regularized gravity action, $S_{\text{reg}}$, is invariant under such diffeomorphisms. This is a consequence of the fact that these diffemorphisms fix (i) and (ii).

# Bulk diffeomorphisms

But the counterterms that are added to make the action finite are *not* invariant. The peccant counterterm is a logarithmically divergent term:

$$\mathcal{L}_{\mathrm{ct}(d)} = \sqrt{g_{(0)}}\; a_{(d)}\; \log \epsilon \; ,$$

where the combination $\sqrt{g_{(0)}}\; a_{(d)}$ is invariant: $\delta(\sqrt{g_{(0)}}\; a_{(d)}) = 0$.
The counterterm then transforms as:

$$\begin{aligned} \delta\mathcal{L}_{\mathrm{ct}(d)} &= \sqrt{g_{(0)}}\; a_{(d)}\, \frac{\delta\epsilon}{\epsilon} = -\sqrt{g_{(0)}}\; a_{(d)}\, \xi =: 8\pi G_{\mathrm{N}} \sqrt{g_{(0)}}\; \mathcal{A}\, \xi \\ \delta W_{\mathrm{ren}} &= -\frac{1}{16\pi G_{\mathrm{N}}} \int \mathrm{d}^d x\, \sqrt{g_{(0)}}\, \mathcal{A}\, \xi \quad , \qquad \mathcal{A} = \text{anomaly} \end{aligned}$$

so that the renormalized generating functional (and the bulk action) depend on the chosen representative of the boundary conformal structure.

Thus in odd bulk dimensions, infrared divergences break part of the bulk diffeomorphisms. Only the bulk diffeomorphisms that do not yield a boundary Weyl transformation are symmetries of the renormalized action.

# Finite ambiguities and boundary conditions

The anomaly comes from the **ambiguity (choice) in separating between finite and divergent terms** in the action. Once such a choice is made, and after transforming the action by a diffeomorphism, additional *finite terms* come from the *divergent part of the action:* so that the finite (renormalized) action effectively changes in an anomalous way.

The finite part of the action is ambiguous for a second reason: **finite boundary counterterms** can always be added without changing the equations of motion (cf. Nester et al.). These finite boundary terms correspond to different boundary conditions for the fields on the boundary. Adding a suitable boundary term, we can go from a Dirichlet to a Neumann boundary problem, where instead of fixing the field on the boundary, we fix its canonically conjugate momentum.

# Finite ambiguities and boundary conditions

Take for example a conformally coupled scalar field in AdS with a cutoff $r=\epsilon$. The usual **Dirichlet boundary problem** is as follows:

$$\begin{aligned} S &= \int_{M_\epsilon} \mathrm{d}^4x\,\sqrt{g}\left(\frac{1}{2}(\partial\Phi)^2-\frac{8}{\ell^2}\,\Phi^2\right)+\frac{1}{2\ell}\int_{\partial M_\epsilon}\mathrm{d}^3x\,\sqrt{\gamma}\,\Phi^2 \\ \Rightarrow\quad \delta S &= (\mathrm{eom})-\frac{1}{\ell}\int \mathrm{d}^3x\;\phi_{(1)}\,\phi_{(0)}\,, \quad \text{where we solved the KG equation in AdS:} \\ \Phi(r,x) &= \frac{r}{\ell}\,\phi_{(0)}(x)+\frac{r^2}{\ell^2}\,\phi_{(1)}(x)+\ldots \quad \text{The expectation value follows from the action:} \\ \langle\mathcal{O}_{\Delta=2}\rangle &\equiv \pi_\Phi := \frac{\delta S}{\delta\phi_{(0)}}=-\frac{1}{\ell}\,\phi_{(1)}\;. \end{aligned}$$

This is the standard AdS-CFT dictionary.

# Finite ambiguities and boundary conditions

We can change the Dirichlet boundary condition into a **Neumann boundary condition** by adding a boundary term to the action (the eom are unchanged):

$$\begin{aligned} \tilde{S} &= S + \frac{1}{\ell}\int \mathrm{d}^3x \ \phi_{(1)}\,\phi_{(0)} \\ \Rightarrow \quad \delta\tilde{S} &= (\mathrm{eom}) + \frac{1}{\ell}\int \mathrm{d}^3x \ \phi_{(0)}\,\delta\phi_{(1)} \equiv 0 \ . \end{aligned}$$

Thus we are holding the sub-leading term, $\phi_{(1)}$, fixed on the boundary.

**Boundary interpretation:** since it is $\phi_{(1)}$ rather than $\phi_{(0)}$ that is being held fixed, $\phi_{(1)}$ is interpreted as a source for an operator $\tilde{\mathcal{O}}(x)$ in the CFT. By dimensional counting, this operator has dimension 1, rather than dimension 2. The expectation value of this operator is calculated, as usual, by taking the functional derivative:

$$\langle \tilde{\mathcal{O}}_{\Delta=1} \rangle = \frac{1}{\ell}\,\phi_{(0)}$$

gives an alternative AdS-CFT dictionary.

# Finite ambiguities and boundary conditions

In this simple case, the two CFT's are each other's Legendre transforms. In the gravity theory, this works because the two modes are actually normalisable, and so their roles can be interchanged.
There is also a mixed boundary problem if we add a boundary term that depends only on $\phi_{(0)}$:

$$
\begin{aligned}
S' &= S + \int \mathrm{d}^3x\,\mathcal{F}(\phi_{(0)}) \\
\Rightarrow\quad \delta S' &= -\frac{1}{\ell}\int \mathrm{d}^3x\,\phi_{(1)}\,\delta\phi_{(0)} + \int \mathrm{d}^3x\,\mathcal{F}'(\phi_{(0)})\,\delta\phi_{(0)} \equiv 0 \\
\Rightarrow\quad \phi_{(1)} &= \ell\,\mathcal{F}'(\phi_{(0)}) \\
\langle \mathcal{O}_{\Delta=2}\rangle &= 0\,.
\end{aligned}
$$

The same can be done for gravity: different boundary terms give rise to different boundary conditions there as well. There an interesting boundary interpretation... but that is a story for another day...

# On background-independence

'There are some claims that string theory does not need a background independent formulation, and can be instead defined for **fixed boundary or asymptotic conditions as dual to a field theory on a fixed background, as in the AdS/CFT correspondence.** To respond to this... it is hard to see how a theory defined only in the presence of boundary or asymptotic conditions, as interesting as that would be, could be taken as a candidate for a complete formulation of a fundamental theory of spacetime. This is because the boundary or asymptotic conditions can only be interpreted physically as standing for the presence of physical degrees of freedom outside the theory... Hence, it seems reasonable to require that a quantum theory of gravity, which is supposed to reproduce general relativity, must also make sense as a theory of a whole universe, as a closed system.' (Smolin 2005, pp. 23-24).

# On background-independence (De Haro 2015)

Background-independence is not a precise term with a fixed meaning (see e.g. Belot (2011), Giulini (2007)). But one can distinguish:

(i) A **minimalist sense of background-independence,** closely modelled on the properties of general relativity. This sense *admits* boundary or initial conditions, which are required to solve Einstein's equations *regardless of the value of the cosmological constant*. It does *not* admit initial or boundary conditions that are not required to solve the dynamical equations of motion.

(ii) An **extended sense:** the initial or boundary conditions also must be *dynamically determined.* One should beware of promoting this to an *a priori* standard for background-independence: for it would render GR background-dependent. Thus throwing the baby out with the bath-water!

The boundary conditions used in standard AdS-CFT are of course **of the type admitted under (i),** i.e. they are *required* by Einstein's equations. **In the minimalist sense, AdS-CFT is fully background-independent.** Perhaps recent developments on $T\bar{T}$-deformations (and other approaches that "look into the bulk") can teach us about (ii)...

# Lee Smolin on background-independence (2018)

Lee Smolin seems to agree, since he now makes a distinction between background structures that are solutions of dynamical equations, and those that are not:

'What I mean when I say that a theory should be background-independent is there is no background structures, such as a geometry in an asymptotic region, which is not a solution to dynamical equations of motion but simply a choice made, an arbitrary choice made in the construction of the theory.'
(Beyond Spacetime summer school, 2018).

# Summary

1. Noether, Klein, and Hilbert on GR: Noether's theorem gives Bianchi *identities* not conservation laws. Einstein: introduce a pseudotensor.
2. Penrose (1982): consider quasilocal quantities. Brown and York (1993) give one such construction (quasilocal stress-energy tensor): not conserved unless there are Killing vectors on the boundary.
3. AdS-CFT:

$$\text{Brown-York stress-energy tensor} \equiv \langle T_{ij} \rangle_{\text{CFT}} \ .$$

4. It needs to be renormalized:

$$\text{bulk diffeo invariance broken} \equiv \text{conformal anomaly in CFT.}$$

5. Finite term ambiguities in the action sometimes give rise to a different AdS-CFT dictionary.

**Thank you!**

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